LAPLACE

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The editor of the MONTHLY has done me the honor to invite me to contribute an article in connection with the bicentenary of Laplace, which falls this year. As Laplace's collected works, which were published under the auspices of the French Academy of Sciences in 1878–1912, occupy fourteen large volumes, it will obviously not be possible to review his discoveries in detail: a mere catalogue of them would more than exhaust the space at my disposal. I do not propose, therefore, to do much more than to give an account of the dramatic experience, very early in his career, when, by solving a problem which had baffled both Euler and Lagrange, he came to be recognized as the greatest living mathematician. I wish to take the opportunity also of correcting some errors with regard to his life and character which have crept into well-known popular histories of mathematics.

When Laplace began his active career as a mathematician, more than eighty years had elapsed since the publication of Newton's *Principia*. For long after its first appearance, that greatest of all works of science had met with considerable opposition. The most eminent mathematicians of the end of the seventeenth century—Huygens, Leibnitz, John Bernoulli, Cassini,—declared against the Newtonian theory of gravitation; even in Cambridge, Newton's own University, natural philosophy continued for long to be studied in the text-book of Rohault, a Cartesian work. It was not until 1745 that the theory of the motions of the heavenly bodies began to be carried beyond the stage it had reached in the *Principia*; and for forty years after that, attention was focused almost exclusively on certain phenomena, which were well attested by astronomical observation, and which seemed to be irreconcilable with the Newtonian theory.

The most striking of these was what was called the great inequality of Jupiter and Saturn. From a comparison of ancient and current observations, it was found that for many centuries the mean motion, or average angular velocity round the sun, of Jupiter, had been continually increasing, while that of Saturn had been continually decreasing. Hence it could be inferred, by Kepler's third law, that the mean distance of Jupiter from the sun was always decreasing, and that of Saturn increasing; so the ultimate fate of Jupiter would be to fall into the sun, while Saturn would wander into space and be lost altogether to the solar system. This of course assumed that the phenomenon was truly secular, that is to say, that it proceeded always in the same direction, with a cumulative effect; but the observational evidence gave no support to any alternative view. Attempts to bring the great inequality of Jupiter and Saturn within the compass of the Newtonian theory of gravitation were made by Euler in 1748 and 1752, and by Lagrange in 1763, but without success. In 1773 Laplace, then aged twenty-four, took the matter up. Carrying the work to a higher degree of approximation than his predecessors, he was surprised to find that in the expression

for the disturbing action of Jupiter upon the mean motion of Saturn, the secular terms cancelled each other out; and following up the clue thus provided, he was able to establish a general theorem, to the effect that the mean motions of the planets cannot have any secular accelerations whatever as a result of their mutual attractions. It seemed as if he, like Euler and Lagrange, had failed to account for the observed fact, and not only so, but had even shown the Newtonian theory to be definitely incompatible with it. However, instead of accepting this consequence, he inferred that the inequality of Jupiter and Saturn was not truly secular, but was periodic, of very long period; and with this new idea, he set to work again, examining the way in which long-period inequalities might arise. Any long-period inequality must arise from a term of the same long period in the perturbing function. Now there were no long-period terms with appreciable coefficients in the perturbing function of one planet or another, so the coefficient of such a term must be very small, and the term might be represented as $p \sin qt$, where p is small, and where q must also be small since the period is long. But Laplace saw that, by the double integration which was involved in the passage from the perturbing function to the expression for the inequality, this term would become $(p/q^2) \sin qt$; and if q were small enough, the coefficient (p/q^2) might be quite large, even though p were very small. This was the true solution of the problem, and all that remained to be done was to identify the small term $p \sin qt$ in the perturbing function which produced the result. Laplace now recalled that five times the mean motion of Saturn is very nearly equal to twice the mean motion of Jupiter; so if n, n' are the mean motions, then $5n-2n'$ is very small, and the term in $\sin [(5n-2n')t]$ in the perturbing function would be of very long period; moreover, its coefficient would involve, as factors, comparatively high powers of the eccentricities and inclinations of the orbits, and so would be very small. Thus it would satisfy all the conditions required. So he arrived finally at a complete explanation of the great inequality; it was not secular, but was periodic, of very long period; and it was due, ultimately, to the fact that the mean motions of the two planets were nearly commensurable. The comparison of his theory with observation left no doubt of its truth.

From the point of view of the solution of the differential equations of motion of the planets, the ratio of the mean motions is classified as one of the arbitrary constants of the solution; and thus Laplace's discovery showed that the nature of the solution of differential equations may be profoundly influenced by the circumstance that one of the arbitrary constants of the solution is a rational or an irrational number. This is the root cause of great difficulties with regard to the convergence of the series of Celestial Mechanics, a subject which calls urgently for further research.*

Laplace's explanation of the great inequality of Jupiter and Saturn was the first of a long series of triumphs, which are recorded in the 2000 pages of his *Traité de Mécanique Céleste*, and which achieved the complete justification of the

* References for the literature may be found in the last chapter of my book *Analytical Dynamics*.

Newtonian theory. He made also many important discoveries in theoretical physics, and indeed he was interested in everything that helped to interpret Nature; pure mathematics for its own sake, however, did not greatly appeal to him, and his contributions to pure mathematics were mostly thrown off as mere by-products of his great works in natural philosophy. Yet there are several cases where a part of one of Laplace's papers, after undergoing a straightforward development at the hands of others, has come to be regarded as an important branch of pure mathematics, to which a substantial volume may fitly be devoted. For instance, E. W. Hobson's great work *The Theory of Spherical and Ellipsoidal Harmonics* (Cambridge, 1931) represents the natural flowering of Laplace's memoir of 1782 on the attraction of spheroids. G. Doetsch's book, *Theorie und Anwendung der Laplace-Transformation* (Berlin, 1937) describes the inevitable evolution of the method of "generating functions" introduced by Laplace in the *Théorie analytique des probabilités*; and any book on the Theory of Determinants consists chiefly of developments of the expansion-theorem which Laplace discovered when he was twenty-three (namely, that any determinant is equal to the sum of all the minors that can be formed from any selected set of its rows, each minor being multiplied by its algebraic complement); for most theorems on determinants can be proved by equating two different Laplace expansions of the same determinant to each other.

When in the course of his researches he comes to a situation where a heavy piece of pure mathematical working is needed, he often says, "Il est facile de voir," and gives the result without saying how he got it. His power of solving problems in pure mathematics has perhaps never been equalled, but he seems to have thought nothing of it, and to have assumed that it was possessed by all the readers of his works.

Coming lastly to personal matters, it is surprising that such learned, and usually well-informed and careful writers as D. E. Smith and Florian Cajori should have fallen into serious error regarding Laplace's origin and early history. The former says, "He was born in poverty, and owed his early education to the interest which his promise excited in men of intellectual power. Of these days of struggle he never spoke. Almost the first reliable records that we have of his life show him studying and afterwards teaching mathematics in the military school at Beaumont"; while Cajori's account is: "Very little is known of his early life. When at the height of his fame, he was loath to speak of his boyhood, spent in poverty. Some rich neighbors who recognized the boy's talent assisted him in securing an education. As an extern he attended the military school at Beaumont, where at an early age he became teacher of Mathematics."

For these statements there is no foundation. Laplace's father, Pierre de Laplace, was Syndic of Beaumont, proprietor of the small estate of Mérisier in the neighborhood, and well-connected, being the nephew of Maitre Olivier de Laplace, Chirurgien Royal. There is no reason to suppose that he was unable or unwilling to pay for the education of his only son Pierre-Simon; and our disbelief becomes complete when we learn that the "military school at Beaumont"

had no existence at that time. The school in which the boy was actually educated between the ages of 7 and 16 was attached to a Benedictine priory at Beaumont, and most of the teachers were Benedictine monks; they employed other teachers however, one of whom seems to have been Pierre-Simon's uncle, Father Louis de Laplace, a secular priest. At the age of 16, Pierre-Simon proceeded to the University of Caen, where he seems to have remained for five years, and where his first mathematical paper was written (it was published in the *Miscell. Taurin.*, then edited by Lagrange).*

Cajori adds "The political career of this eminent scientist was stained by servility and suppleness." It is difficult to see what justification there is for this statement. Laplace placed his scientific knowledge and ability at the disposal of whatever Government was in power; he was President of the Bureau des longitudes, President of the Commission de réorganisation de l'École polytechnique, and so forth; and was highly esteemed both by Napoleon, who made him a Count, and by the restored Bourbons, who further promoted him to be a Marquis. It is unnecessary to postulate "servility and suppleness" in order to account for the fact that he was *persona grata* to the successive rulers of France; his eminence as the greatest living man of science, and his value as an organizer of scientific education and research, made them glad to enlist his services, and to reward him with dignities never bestowed before nor since on a mathematician.

EQUIVALENCE RELATIONS IN ALGEBRAIC SYSTEMS

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1. Introduction. This note is addressed primarily to an interested reader of modern algebra and its purpose is to present an elementary exposition of equivalence relations including a variety of illustrations, along with applications to group theory.

Although the concept of an equivalence relation is a prerequisite for a careful phrasing as well as a full understanding of a great variety of statements in mathematics, frequently the notion as such is met for the first time in an introductory text on modern algebra [see, for example, references **2, 3**]. Its inclusion in such a text is a certainty due to the usefulness of the notion in clarifying the intent of many definitions and in developing efficiently fundamental results for groups and other algebraic systems. However, since its applicability is not restricted to algebra, it seems worth while to examine the concept independently.

* The facts regarding Laplace's family, school, and University, are fully discussed in an article by Professor Karl Pearson in Biometrika 21 (1929), 202.

2. Equivalence relations. A binary relation $\mathfrak{R}$ over a set $S=\{a, b, \cdots\}$ is any function $\mathfrak{R}(a, b)$, where a, b range independently over S, with values in the set consisting of two elements: true, false. In the event $\mathfrak{R}(a, b)=$ true, we write

$$a\mathfrak{R}b,$$

and in the contrary case a solidus is superimposed upon the $\mathfrak{R}$.

A binary relation $\mathfrak{R}$ over S is called an *equivalence relation* if it satisfies the axioms,

(1) Reflexitivity: $a\mathfrak{R}a$ for every a in S.

(2) Symmetry: $a\mathfrak{R}b$ implies $b\mathfrak{R}a$.

(3) Transitivity: $a\mathfrak{R}b$, $b\mathfrak{R}c$ imply $a\mathfrak{R}c$.

For a characterization of equivalence relations another definition is needed. A partition $P=P(C_\alpha)$ of S into classes C_α is a collection of non-empty subsets C_α such that each element of S belongs to one and only one subset.

THEOREM 1: *An equivalence relation $\mathfrak{R}$ over the set S defines a partition $P(C_\alpha)$ of S where the class C_a containing a consists of all x in S such that $x\mathfrak{R}a$. Conversely, a partition of S defines an equivalence $\mathfrak{R}$ where $a\mathfrak{R}b$ if and only if a and b belong to the same class.*

Proof: By (1), a is a member of C_a, so that every element is in at least one class. But if a is in both C_b and C_d, so that $a\mathfrak{R}b$ and $a\mathfrak{R}d$, (2) implies $b\mathfrak{R}a$ and then using (3), $b\mathfrak{R}d$. Hence $C_b=C_d$ and a is in only one class.

For the converse, it is clear that (1), (2), (3) are satisfied.

In the set of integers the familiar notions of $a\,|\,b$ (a divides b), $a<b$, $(a, b)=1$ (a is relatively prime to b), and $|a-b|<1$ are examples of binary relations satisfying one or more of (1), (2), (3). We list next several relations that are actually equivalences.

E_1. The relation $a\equiv b \pmod{n}$ in the set of integers. This example is responsible for, in the general case, the notation $a\equiv b(\mathfrak{R})$ in place of $a\mathfrak{R}b$, and the phrase "a residue class mod $\mathfrak{R}$" to denote a class C_a.

E_2. The relation $f(x)=g(x)$, for almost all $x\in I$, in the set of all real, single-valued functions $f(x)$ defined over an interval I.

E_3. The relation A similar to B in the set of all n by n matrices with entries from a field.

E_4. The relation T_α similar to T_β in the set of all plane triangles.

E_5. The relation A is obtainable from B by a rotation and translation, in the set of all proper conic sections.

E_6. This example, along with several definitions arising from special cases of it, is included for future reference. Let G and G' denote two sets and σ a single-valued correspondence from G onto G', *i.e.*, σ is a rule associating with each element $g\in G$, a unique element $g'=\sigma(g)\in G'$ such that $\sigma(x)=g'$ has a solution $x\in G$ for every $g'\in G$. Then $g_1\equiv g_2$, if and only if $\sigma(g_1)=\sigma(g_2)$, is an equivalence relation in G.

In the event $\sigma(x)=g'$ always has a unique solution for all $g'\in G'$, σ is a one-one correspondence between G and G'. If G and G' are multiplicative systems, *e.g.*, groups, and σ has the property $\sigma(ab)=\sigma(a)\sigma(b)$, then σ is called a homomorphism and G' a homomorphic image of G. If, moreover, σ is one-one, the homomorphism is called an isomorphism.

E_7. Our final example, the equality relation, is in a sense the most fundamental since in its analysis we arrive at the postulates assumed for an equivalence relation. Indeed, if one reflects on the everyday usage of this relation as a synonym for either identity or a qualified degree of likeness, *e.g.*, when we say two triangles are "equal," we may have in mind only the agreement of the areas, one soon comes to the conclusion that equality satisfies axioms (1) through (3). On the other hand, Theorem 1 demonstrates that these are sufficient to assure the desired separation into classes, and hence that they characterize equality. As such, any equivalence relation may be called an "equals" relation.

3. A fundamental application. The last example above leads immediately to one fundamental application of equivalence relations. On one hand we have seen that the equivalence axioms characterize equality. On the other hand, when dealing with a concrete mathematical system, we invariably have as an inherent part of the system a rule for deciding whether two elements are one and the same, or whether they are distinct, *i.e.*, we have given an equality. Thus when one realizes the duplication in effort that frequently arises in the analysis of various concrete systems and decides to study an abstract system whose properties are those underlying various concrete systems, one such assumed property will be the existence of an equivalence relation. Usually this basic relation is denoted by $=$, to distinguish it from further equivalences that may be defined.

Invariably at some stage of an investigation of a mathematical system S, one wishes to identify elements which although unequal, exhibit a certain likeness which we symbolize by $\mathcal{L}$. Any suitable criterion for indiscernibility must lead to a partition of S; hence $\mathcal{L}$ must be an equivalence relation. That is, $\mathcal{L}$ must measure up to the equivalence axioms to serve the purpose intended for it. Each of the examples E_1 through E_6 exhibits a likeness $\mathcal{L}$ in the set at hand. An equivalence class mod $\mathcal{L}$ consists of all elements of S indiscernible with respect to $\mathcal{L}$. With the way cleared for emphasis upon equivalence classes, it is sometimes in order to regard these classes as elements of a new set Σ and assign them names. For example, recall the definition of a cardinal number. In transfinite arithmetic one calls two sets equivalent if and only if they can be put in one-one correspondence, and then defines the cardinal number of a set S as the set of all classes which are equivalent with S. Accompanying the transition from S to Σ is the replacement of equivalence mod $\mathcal{L}$ by equality (identity) of elements in Σ. The reverse step is frequently used in computations: calculations with elements of Σ are performed mod $\mathcal{L}$ with elements of S chosen from the Σ-elements in question (*i.e.* so-called representatives of the equivalence classes). Calculations with rational numbers serve as an example of this.

4. A restricted type of equivalence relation. The minimum requirements for an algebraic system certainly include the existence of a set S with an equality relation for which there is defined a binary law of composition, *i.e.* a single-valued function $a \cdot b$ (or simply ab) of pairs a, b such that $a \cdot b$ is in S for a, b in S. Adopting this as our starting point, we superimpose an equivalence $\mathfrak{R}$ on S in order to point out how one is led to an important restricted type of equivalence relation. Namely, denoting by Σ the set of equivalence classes C_a mod $\mathfrak{R}$, we raise this question: Can an operation $\odot$ be defined in Σ based upon the operation $\cdot$ in S? We proceed along the lines of what might be one's first attempt* to investigate this question by tentatively defining

$$C_a \odot C_b = C_{a \cdot b}, \tag{4}$$

thus apparently making the product dependent upon the choice of class representatives. This difficulty is overcome if

$$C_{a'} = C_a \quad \text{and} \quad C_{b'} = C_b \quad \text{imply} \quad C_{a' \cdot b'} = C_{a \cdot b}$$

or, what amounts to the same thing,

$$a' \mathfrak{R} a \quad \text{and} \quad b' \mathfrak{R} b \quad \text{imply} \quad a'b' \mathfrak{R} ab. \tag{5}$$

An equivalent form of (2) is

$$a' \mathfrak{R} a \quad \text{implies} \quad a'x \mathfrak{R} ax \quad \text{and} \quad xa' \mathfrak{R} xa \quad \text{for all } x. \tag{6}$$

This sufficient condition is also a necessary one upon $\mathfrak{R}$ in order that (4) be a well-defined operation. Equivalences satisfying (6) will be called regular.

Then if $\mathfrak{R}$ is regular we observe that the correspondence ϕ of S onto Σ defined by

$$\phi(x) = C_a \quad \text{if and only if} \quad x \in C_a, \tag{7}$$

is a homomorphism (see E_6). For if along with $\phi(x) = C_a$, $\phi(y) = C_b$ then $\phi(xy) = C_{xy} = C_x \odot C_y$ using (4). But since $C_x = C_a$ and $C_y = C_b$ we have finally $\phi(xy) = C_a \odot C_b$. The homomorphism (7) is usually referred to as the "homomorphism of inclusion" or the "natural homomorphism."

In passing, we notice that if $\mathfrak{R}$ is the equality relation in S the above has the following significance. We take as the elements of our algebraic system the equivalence classes mod $\mathfrak{R}$ and assume that R is regular with repect to $\cdot$. Then (4) is a well-defined operation. Thus, a careful definition of a group begins with the existence of a set S, a binary operation $(\cdot)$, an equivalence $(=)$, regular with respect to $\cdot$. The group elements are then the equivalence classes mod $=$.

* As an alternative "stimulus" for (4) we might take the following as our criterion for multiplication in Σ: Σ should be a homomorphic image of S under a correspondence mapping all elements of S belonging to an equivalence class onto an element of Σ. But the existence of such a homomorphism immediately implies the existence of one mapping the class containing a upon C_a and the homomorphism property then requires that (4) hold.

To illustrate this point, let S denote the set of all plane rotations ρ of a given equilateral triangle into itself. If $\mathfrak{R}$ is defined by $\rho\mathfrak{R}\rho'$ if and only if the number of degrees in both agree, we obtain an infinite cyclic group. Whereas if $\mathfrak{R}$ is defined by $\rho\mathfrak{R}\rho'$ if and only if they produce the same permutation of the vertices, a cyclic group of order three results.

5. Further applications to groups. Returning to the central idea of the preceding section, let us specialize our system to a group G with a regular equivalence $\mathfrak{R}$. Then, defining an operation in $\Gamma = \{C_a, C_b, \cdots\}$ by (4) gives a homomorphic image of G. But it is easily seen that a homomorphic image of a group is necessarily a group. Hence Γ is a group; its unit element is C_e where e is the unit in G. Thus, shrinking a group with the aid of a regular equivalence produces a homomorphic image.

Conversely, given a homomorphic image Γ of G, a partition, and therefore an equivalence $\mathfrak{R}$, is defined in G (see E_6) and the homomorphism property immediately demonstates that $\mathfrak{R}$ is regular. Hence the problem of finding all homomorphic images of G amounts to that of finding all regular equivalences over G.

Concerning this problem we first mention the coset decomposition of a group G with respect to a subgroup H. Define

$$a\mathfrak{R}b \quad \text{if and only if} \quad a = hb,\ h \in H.$$

Since $a = ae$, $a\mathfrak{R}a$; if $a = hb$ then $b = h^{-1}a$, so that $\mathfrak{R}$ is symmetric. Finally $a = hb$, $b = h'c$ implies $a = (hh')c$, so that $\mathfrak{R}$ is transitive and hence an equivalence. The equivalence class C_a is Ha, a so-called right coset of H, and

$$G = \Sigma Ha.$$

Since $a\mathfrak{R}b$ implies $ax\mathfrak{R}bx$, $\mathfrak{R}$ is right regular, *i.e.*, satisfies the first part of (6).

But conversely, starting with a right regular equivalence $\mathfrak{R}$ in G we find that C_e is a subgroup and $C_e = C_e a$, since $b\mathfrak{R}a$ implies $b_a{}^{-1}\mathfrak{R}e$; hence $ba^{-1} \in C_e$, $b \in C_e a$ and conversely. Hence the various right regular equivalences in G are determined by the right coset decompositions of G with respect to its subgroups. It is of interest to note that once the elements equivalent to the unit e are known in a right regular equivalence, the entire decomposition of G is known.

Similar results are obtained for left regular equivalences and we can now analyze the regular ones. If $\mathfrak{R}$ is regular, then on one hand it defines the left coset decomposition with respect to the subgroup H of all elements x such that $x\mathfrak{R}e$. On the other hand it defines a right coset decomposition with respect to the same subgroup. Hence $\mathfrak{R}$ stems from a subgroup for which the left cosets are identical with its right cosets. Such a subgroup N we call normal, and have

$$xN = Nx \quad \text{for all} \quad x \in G.$$

Clearly this property characterizes the normal subgroups. Conversely, a normal subgroup defines a regular equivalence.

If N is a normal subgroup of G we now know that its cosets xN form a group with multiplication rule (4):

$$aN \cdot bN = abN$$

and unit N. The resulting (factor) group, G/N, is homomorphic to G and conversely every homomorphic image of G can be duplicated by (*i.e.* is isomorphic to) such a factor group.

The analysis of a ring along the same lines as above yields the corresponding homomorphism theorem just as efficiently.

6. Conclusion. In conclusion we mention that the set E of all possible equivalence relations definable over a given set S has been made an object of study [1, 4]. If operations are defined appropriately, E becomes an example of a lattice.

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AN APPLICATION OF NEWTON'S POWER-SUM FORMULAS

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1. Introduction. If $f(x)$ and $g(x)$ are real polynomials, then we may ask under what conditions

$$f(g(x)) = g(f(x)) \tag{1}$$

identically in x. Without further restrictions, there are clearly an infinite number of $g(x)$'s which satisfy (1) for a fixed $f(x)$, as for example all the polynomials of type $f(\cdots(f(x))\cdots)$, where the operator f is iterated n times. The main purpose of the present note is to prove in an elementary way that for a given $f(x)$ the only $g(x)$'s having the same degree as $f(x)$ and satisfying (1) are: $g(x) = f(x)$ if the degree is even, and one other possibility if the degree is odd and greater than one (see Theorem I). An application of this result is made to obtain certain uniqueness properties of families of mutually commutative polynomial operators. For the general properties of polynomial substitution one may consult the papers listed in the references.

2. The main theorem. Let f and g refer to the polynomial operators defined by

$$f(x) = \sum_{i=0}^{n} a_i x^{n-i} \quad \text{and} \quad g(x) = \sum_{i=0}^{n} b_i x^{n-i},$$

where the coefficients are real. Also, let fg mean the operator defined by the substitution $f(g(x))$. Then we can prove

THEOREM I: *If f and g are real polynomial operators of equal degree $n>1$, then $fg=gf$ holds if and only if $f=g$, unless $f(x)$ is of the form $a_0(x-c)^{2k+1}+b_1(x-c)^{2k-1}+\cdots+b_{2k}(x-c)+c$, in which case $g(x)$ may also be $-f(x)+2c$, where $c=-a_1/na_0$, $n=2k+1$.*

Geometrically this means that if $y=f(x)$ is symmetric to a point on $y=x$, then and only then does it have a non-trivial commuting polynomial of the same degree. If $n=1$, each point on $y=x$ defines an infinite bundle of commuting linear functions.

Proof. Let r_i and s_i, $i=1, \cdots, n$ be the zeros of $f(x)$ and $g(x)$ respectively. Then $fg=gf$ implies, as a necessary condition,

$$a_0\prod_{i=1}^{n}(g(x)-r_i)=b_0\prod_{i=1}^{n}(f(x)-s_i).$$

Thus the zeros of $f(g(x))$ are distributed in two ways, once among the factors $\{g(x)-r_i\}$ and again among the factors $\{f(x)-s_i\}$. This distribution can be effectively utilized by application of Newton's power-sum formulas. Let σ_i and S_i, $i=1, \cdots, n$ denote the usual elementary symmetric functions and the power sums of the roots, respectively. Then Newton's formulas can be expressed as follows:

$$\begin{pmatrix} 1 & & & & \\ -\sigma_1 & 1 & & & \\ \sigma_2 & -\sigma_1 & 1 & & \\ \vdots & & \ddots & \ddots & \\ (-1)^{n-1}\sigma_{n-1} & \cdots & \cdots & \cdots & 1 \end{pmatrix}\begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_n \end{pmatrix}=\begin{pmatrix} \sigma_1 \\ -2\sigma_2 \\ 3\sigma_3 \\ \vdots \\ (-1)^{n-1}n\sigma_n \end{pmatrix}.$$

These may be applied to each factor and added over the factors. Two expressions for each S_i of $f(g(x))$ will thus be generated, and by equating them the desired relations among the a_i and b_i appear. The first two steps and the critical last step (for $n>3$) are as follows:

$$S_1=\sum_{i=1}^{n}(-b_1/b_0)=\sum_{i=1}^{n}(-a_1/a_0), \quad \text{implying} \quad a_1/a_0=b_1/b_0.$$

$$S_2=\sum_{i=1}^{n}[(-b_1/b_0)^2-2b_2/b_0]=\sum_{i=1}^{n}[(-a_1/a_0)^2-2a_2/a_0]; \quad \text{hence} \quad a_2/a_0=b_2/b_0.$$

. . . . . .

$$S_n=\sum_{i=1}^{n}[\cdots+(-1)^{n-1}n(b_n-r_i)/b_0]=\sum_{i=1}^{n}[\cdots+(-1)^{n-1}n(a_n-s_i)/a_0];$$

hence $(1/b_0)\sum(b_n - r_i) = (1/a_0)\sum(a_n - s_i)$; and finally $n(a_nb_0 - b_na_0) = a_1 - b_1$.

Now, an examination of the condition $fg=gf$ gives immediately that $a_0b_0^n = a_0^nb_0$. So if n is even, then $a_0=b_0$; otherwise $a_0=\pm b_0$. Hence

(a). For $a_0=b_0$, we get $a_i=b_i$ for $i=1, \cdots, n-1$. Furthermore, $na_0(a_n-b_n) = 0$, or $a_n=b_n$. These are obviously sufficient conditions.

(b). For $a_0=-b_0$, we get $a_i=-b_i$ for $i=1, \cdots, n-1$, and $na_0(a_n+b_n) = -2a_1$, or $b_n=-a_n-2a_1/na_0$. That is, $g(x)$ differs from $-f(x)$ by a constant. The implication of this necessary condition is clear if one writes $g(x)=-f(x)+k$ and substitutes in $fg=gf$. For then $f(-f(x)+k)=-f(f(x))+k$, or $f(k-y) = -f(y)+k$. Setting $y=z-k/2$ shows that $f(x)$ must be symmetric to $(k/2, k/2)$. Such a function will be of the form given in the statement of the theorem, and a direct check shows that it commutes with $-f(x)+2c$.

3. Application. As an application of Theorem I, we may show

THEOREM II. *Any family F of real mutually commutative polynomial operators, with at least one operator of each degree,* has exactly one of each degree. The prime degree operators determine the whole family.*

This is a corollary of the

LEMMA. *If F_0 is any family of real mutually commutative polynomial operators containing at least one operator of even degree, then F_0 contains only one operator of each degree represented.*

Proof. Let e be the degree of the even operator f_e which is in F_0 by hypothesis. Then, by Theorem I, f_e is the only operator of degree e in F_0, this being true for any even degree. Let f_a and g_a be of degree a and in F_0. Then f_af_e and g_af_e commute with one another, and being of even degree must be equal. Hence $f_a=g_a$. This proves the Lemma.

In particular, the Lemma implies that in F of Theorem II there is just one operator of each degree. If f_2 in F is given, then f_2^n is in F for all n, where f^n means f iterated n times. In fact, when the prime degree operators f_{p_i} are given, then for $a=\prod p_i^{a_i}$, the composite $\prod f_{p_i}^{a_i}$ is in F. Hence $f_a=\prod f_{p_i}^{a_i}$. This proves Theorem II.

Perhaps the best known family of type F are the Tschebycheff polynomials.

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* A study of such families has been made by H. Thielman and H. Block.

THE COMPOSITION OF QUADRATIC BINARY FORMS*

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1. Introduction. Though the general basis for the composition of forms lies in the ideal theory, the composition of binary quadratic forms was first developed independent of such theory; in fact, it antedated the existence of ideal theory. Dedekind† laid the basis for the relationship between the composition of ideals and that of forms and Weber‡ developed it. But neither used it to prove Gauss's theorem on duplication nor for other important results in the theory of binary forms. The literature in general seems somewhat hazy on the precise details of the subject. It seems worthwhile therefore to give a self-contained elementary development of the theory assuming neither knowledge of ideal theory nor of quadratic forms. It will be seen that the use of ideal theory adds elegance to the treatment and gives promise of further application.

We first define and develop some of the properties of ideals in a quadratic field. Let $\mathfrak{F}$ be a field obtained by adjoining $\sqrt{\Delta}$ to the field of rational numbers where Δ is a non-square integer congruent to 1 or 0 (mod 4); that is, $\mathfrak{F}$ consists of all numbers of the form $a+b\sqrt{\Delta}$ where a and b are rational numbers. Let $\sigma=\frac{1}{2}(\sqrt{\Delta}+1)$ or $\frac{1}{2}\sqrt{\Delta}$ according as $\Delta\equiv 1$ or 0 (mod 4) and call quadratic integers those numbers expressible in the form $x+y\sigma$ where x and y are rational integers. (We reserve the name "integer" for rational integers.) The set of all quadratic integers in $\mathfrak{F}$ we denote by $J(\Delta)$ or J. It is easy to see that J is closed under addition, subtraction and multiplication; that is, the sum, difference and product of any two quadratic integers are quadratic integers. If α is an element of J we call the number obtained from α by replacing $\sqrt{\Delta}$ by $-\sqrt{\Delta}$ its conjugate and denote the conjugate of α by α^c. Call $\alpha\alpha^c=N(\alpha)$, the norm of α, and see that $N(\alpha)N(\beta)=N(\alpha\beta)$. If α is a quadratic integer, $N(\alpha)$ is an integer.

If now $\alpha_1, \alpha_2, \cdots, \alpha_n$ is a set of numbers of J, we call all numbers $\tau_1\alpha_1+\tau_2\alpha_2+\cdots+\tau_n\alpha_n$ where the τ_i range over all numbers of J, the ideal $\mathfrak{J}=(\alpha_1, \alpha_2, \cdots, \alpha_n)$. If the α_i are such that every number in $\mathfrak{J}$ is expressible uniquely in the form $x_1\alpha_1+x_2\alpha_2+\cdots+x_n\alpha_n$ where the x_i are integers, we call $\alpha_1, \alpha_2, \cdots, \alpha_n$ a basis of $\mathfrak{J}$ and write $\mathfrak{J}=[\alpha_1, \alpha_2, \cdots, \alpha_n]$ with brackets instead of parentheses. Two ideals are said to be equal if they contain the same numbers. An ideal consisting of all numbers $\alpha\rho$, where α is a fixed number of J and ρ ranges over all numbers of J is called a principal ideal and is written (α). If $\mathfrak{J}=[\alpha_1, \alpha_2]$ we call $\mathfrak{J}^c$ the ideal $[\alpha_1^c, \alpha_2^c]$ and $u\mathfrak{J}$ the ideal $[u\alpha_1, u\alpha_2]$.

2. Some properties of quadratic ideals. We first need some classical results on ideals.

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† Dirichlet-Dedekind, Zahlentheorie, ed. 2 (1871), Suppl. XI pp. 488–497.

‡ Heinrich Weber, Lehrbuch der Algebra, ed. 2 (1908), pp. 368–397.

THEOREM 1. *Every ideal $\mathfrak{J}$ of J has a basis $[r, s+u\sigma]$ where r, s, u are integers, r and u both being positive. Furthermore, r is the g.c.d. of the integers in $\mathfrak{J}$ and u is the g.c.d. of the coefficients of σ in the quadratic integers of $\mathfrak{J}$. (Hence r and u are uniquely determined by $\mathfrak{J}$.)*

To prove this, first notice that $\mathfrak{J}$ contains an integer since if α is a quadratic integer in $\mathfrak{J}$, then $N(\alpha)$ is also in $\mathfrak{J}$. If then we let r be the least positive integer in $\mathfrak{J}$ we see that it must divide all integers in $\mathfrak{J}$ since if it does not divide an integer b in $\mathfrak{J}$, then $b=qr+r'$, where r' is a positive integer less than r; b and qr being in $\mathfrak{J}$ implies that r' is in $\mathfrak{J}$ contrary to the supposition that r has least positive value. By the same reasoning we can show that u has the properties desired.

It remains to show that every number of $\mathfrak{J}$ is a unique linear combination of r and $s+u\sigma$ with integer coefficients. To this end let $a+b\sigma$ be a number of $\mathfrak{J}$. Then $b=ux$ determines an integer x and $a+b\sigma-x(s+u\sigma)=a-xs$ is an integer in $\mathfrak{J}$ and hence is equal to ry for some integer y; thus $a+b\sigma=yr+x(s+u\sigma)$. Furthermore $yr+x(s+u\sigma)=y'r+x'(s+u\sigma)$ implies that $x=x'$, $y=y'$ and the representation in terms of the basis is unique. This completes the proof of the theorem.

There are certain restrictions on r, s and u of the theorem above if r, $s+u\sigma$ are to form a basis for an ideal. For instance, 3 and $1+2\sqrt{3}$ do not form a basis of an ideal since the equation $(1-2\sqrt{3})(1+2\sqrt{3})=-11$ shows that -11 is in the ideal $(3, 1+2\sqrt{3})$ while -11 is not expressible in the form $3x+(1+2\sqrt{3})y$ for integers x and y. We now find necessary and sufficient conditions on integers r, s and u with $ru\neq 0$ that $(r, s+u\sigma)$ shall form a basis for an ideal. Suppose first that $(r, s+u\sigma)$ is such a basis. Then all numbers of the ideal are of the form

$$t = r(x_1 + y_1\sigma) + (s + u\sigma)(x_2 + y_2\sigma^e)$$

for integers x_1, x_2, y_1, y_2. The coefficient of σ in t is $ry_1+ux_2-sy_2$ and since, from the theory of numbers, y_1, x_2 and y_2 can be chosen so that this coefficient is the g.c.d. of r, u and s, we see that u, being by Theorem 1 a divisor of all coefficients of σ in numbers of $\mathfrak{J}$, must divide the g.c.d. of r, u and s; hence the equations $r=au$, $s=eu$ determine integers a and e. If, on the other hand, we choose y_1, x_2 and y_2 so that $ry_1+ux_2-sy_2=0$, that is, $ay_1+x_2-ey_2=0$, and let $\sigma+\sigma^e=\epsilon$ where ϵ is 1 or 0 according as $\Delta\equiv 1$ or 0 (mod 4) we have

$$\begin{aligned} t &= rx_1 - (rs/u)y_1 + y_2k/u \\ k &= s^2 + su\epsilon + u^2\sigma\sigma^e = N(s + u\sigma). \end{aligned} \tag{1}$$

Since t is an integer in $\mathfrak{J}$ it must, by Theorem 1, be divisible by r for all values of x_1, y_1 and y_2; hence $k\equiv 0 \pmod{ru}$. Thus we have shown that if $(r, s+u\sigma)$ is a basis under the restrictions of Theorem 1, then r and s are divisible by u and $N(s+u\sigma)$ is divisible by ru. Conversely if these conditions hold we see by reference to the above that u divides all coefficients of σ for numbers of $\mathfrak{J}$ and that,

since all integers in $\mathfrak{J}$ are expressible in the form of t in (1), r divides all integers in $\mathfrak{J}$. Hence we have proved

THEOREM 2. *$\mathfrak{J}=[r, s+u\sigma]$ for r, s and u integral and $ru\neq 0$ if and only if $r=ua$, $s=ue$ determine integers a and e and $N(s+u\sigma)\equiv 0 \pmod{ru}$. That is, we can write $\mathfrak{J}=[au, eu+u\sigma]=u[a, e+\sigma]$ with $N(e+\sigma)\equiv 0 \pmod{a}$.*

We call a basis $[r, s+u\sigma]$ of an ideal a *reduced basis* if r, s and u are integers, r and u both positive. Theorem 2 gives an alternative way of writing a reduced basis.

Suppose $[\alpha_1, \alpha_2]$ and $[\beta_1, \beta_2]$ are two equal or distinct ideals for which the following equations hold:

$$\begin{aligned}\alpha_1 &= \beta_1 t_{11} + \beta_2 t_{21} \\ \alpha_2 &= \beta_1 t_{12} + \beta_2 t_{22}.\end{aligned} \tag{2}$$

Then we may say that the basis $[\alpha_1, \alpha_2]$ is taken into the basis $[\beta_1, \beta_2]$ by the linear transformation T whose matrix is

$$T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}.$$

If the elements of T are integers and its determinant ± 1, we call the transformation unimodular; if its determinant is $+1$ we call it properly unimodular. Considering (α_1, α_2) and (β_1, β_2) as two-row matrices the equations (2) may be written in matric notation as follows:

$$(\alpha_1, \alpha_2) = (\beta_1, \beta_2)T.$$

Next we prove

THEOREM 3. *Two ideals $[\alpha_1, \alpha_2]$ and $[\beta_1, \beta_2]$ are equal if and only if there is a unimodular transformation taking one basis into the other.*

If there is a unimodular transformation taking one into the other the numbers in the ideals are the same, since (2) and T unimodular implies that $x\alpha_1+y\alpha_2 = (xt_{11}+yt_{12})\beta_1+(xt_{21}+yt_{22})\beta_2$ and if the coefficients of α_1 and α_2 are integers, so are the coefficients of β_1 and β_2. Suppose on the other hand, that the ideals are equal. Then there are matrices T and T' with integer elements such that the matric equations $(\alpha_1, \alpha_2)=(\beta_1, \beta_2)T$ and $(\beta_1, \beta_2)=(\alpha_1, \alpha_2)T'$ hold. Then $(\alpha_1, \alpha_2) = (\alpha_1, \alpha_2)T'T$ and since the representation of α_1 and α_2 by α_1 and α_2 is unique we have $T'T=I$, the identity matrix, which shows that $|T'||T|=1$ and hence $|T|=\pm 1$. Thus the theorem is proved.

If $\mathfrak{J}$ has the reduced basis $[ua, ue+u\sigma]$ we define the norm of $\mathfrak{J}$, denoted by $N(\mathfrak{J})$, to be ru and see that $N(\mathfrak{J})=u^2a$. If $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are two ideals in J, their product $\mathfrak{J}_1\mathfrak{J}_2=\mathfrak{J}_3$ is defined to be that ideal consisting of all numbers in J expressible in the form $\sum \rho_i\alpha_i\beta_i$ where the α_i are in $\mathfrak{J}_1$, the β_i are in $\mathfrak{J}_2$ and the ρ_i are quadratic integers. Since the set of numbers expressible in the form

$\sum \rho_i \alpha_i^c \beta_i^c$ coincides with the conjugates of the set of numbers expressible in the form $\sum \rho_i \alpha_i \beta_i$ we see that $\mathfrak{J}_1 \mathfrak{J}_2 = \mathfrak{J}_3$ implies $\mathfrak{J}_1^c \mathfrak{J}_2^c = \mathfrak{J}_3^c$. Multiplication of ideals is associative and commutative since quadratic integers have these properties. We call an ideal $\mathfrak{J} = [ua, ue+u\sigma]$ primitive if 1 is the g.c.d. of a, $2e+\sigma+\sigma^c$, and c where $c = N(e+\sigma)/a$. The following important result holds even if the ideals concerned are neither primitive nor principal but the general proof is much more difficult and the more restricted result is sufficient for our purposes here.

THEOREM 4. *If each of the ideals* $\mathfrak{J}_1$, $\mathfrak{J}_2$ *is either primitive in* J *or principal, then* $N(\mathfrak{J}_1 \mathfrak{J}_2) = N(\mathfrak{J}_1) N(\mathfrak{J}_2)$.

We prove this theorem by showing that it follows from Lemma 1 below. This consequence is direct since, using the lemma, $(N(\mathfrak{J}_1))(N(\mathfrak{J}_2)) = \mathfrak{J}_1 \mathfrak{J}_1^c \mathfrak{J}_2 \mathfrak{J}_2^c = \mathfrak{J}_3 \mathfrak{J}_3^c = (N(\mathfrak{J}_3))$, where $\mathfrak{J}_3 = \mathfrak{J}_1 \mathfrak{J}_2$, and hence $N(\mathfrak{J}_1) N(\mathfrak{J}_2) = N(\mathfrak{J}_3)$. It remains to show

LEMMA 1. *If the ideal* $\mathfrak{J}$ *is primitive or principal, then* $\mathfrak{J} \cdot \mathfrak{J}^c$ *is the principal ideal consisting of all multiples of* $N(\mathfrak{J})$ *by numbers of* J.

To prove this first for $\mathfrak{J}$ primitive, write $\mathfrak{J} = [ua, ue+u\sigma]$ and see that all elements of $\mathfrak{J} \cdot \mathfrak{J}^c$ are linear combinations of u^2a^2, $u^2a(e+\sigma)$, $u^2a(e+\sigma^c)$, $u^2aN(e+\sigma)/a = u^2ac$, with integer coefficients. Thus all numbers of $\mathfrak{J} \cdot \mathfrak{J}^c$ are divisible by u^2a. Furthermore, addition of the second and third numbers above, shows that $\mathfrak{J} \cdot \mathfrak{J}^c$ contains u^2ag where g is the g.c.d. of a, $2e+\sigma+\sigma^c$ and c which, by the primitiveness of $\mathfrak{J}$, is equal to 1. This shows that $\mathfrak{J} \cdot \mathfrak{J}^c$ is the principal ideal (u^2a).

If, on the other hand, $\mathfrak{J}$ is principal it may be written in the form $u(v+w\sigma)$ where v and w are relatively prime. Then $\mathfrak{J} \cdot \mathfrak{J}^c = (u^2 N(v+w\sigma))$ and it remains to show that $N(\mathfrak{J}) = u^2 N(v+w\sigma)$. We do this by finding a reduced basis for $\mathfrak{J}_1 = (v+w\sigma)$. Now every number of $\mathfrak{J}_1$ is expressible in the form $L = (v+w\sigma)(x+y\sigma^c)$ for integers x and y. But $L = vx + wyN(\sigma) + (\sigma+\sigma^c)yv + (wx-yv)\sigma$ shows that x and y may be chosen so that the coefficient of σ is 1. All values of x and y which make $wx - yv = 0$ are integral multiples of $x = v$ and $y = w$. Hence all integers in $\mathfrak{J}_1$ are integral multiples of $v^2 + w^2 N(\sigma) + (\sigma+\sigma^c)vw = N(v+w\sigma)$ which is therefore the norm of $\mathfrak{J}_1$.

It is convenient to call $d(\mathfrak{J})$, the determinant of an ideal $[\alpha_1, \alpha_2]$, the expression $-(\alpha_1^c \alpha_2 - \alpha_2^c \alpha_1)^2/4$. To justify this definition by showing that its value is independent of the particular basis chosen let $(\alpha_1', \alpha_2') = (\alpha_1, \alpha_2) T$ where T is unimodular. Then

$$\begin{vmatrix} \alpha_1' & \alpha_2' \\ \alpha_1'^c & \alpha_2'^c \end{vmatrix} = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \alpha_1^c & \alpha_2^c \end{vmatrix} |T|$$

which implies that the two determinants are equal except perhaps for sign, and hence that the change of basis does not alter $d(\mathfrak{J})$. Furthermore if $\mathfrak{J} = [r, s+u\sigma]$,

where r, s and u are the properties imposed in Theorem 1, we have $d(\mathfrak{J}) = -r^2u^2(\sigma-\sigma^c)^2/4 = -N^2(\mathfrak{J})(\sigma-\sigma^c)^2/4 = -N^2(\mathfrak{J})\Delta/4$.

3. A correspondence between ideal classes and classes of quadratic forms. We first need some of the terminology of quadratic forms. Let the binary form be denoted by $f=ax^2+bxy+cy^2$, where a, b and c are integers. The matrix

$$F = \begin{bmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{bmatrix}$$

we call the matrix of f and the determinant of F, $ac-\frac{1}{4}b^2$, is called the determinant of f and is denoted by $d(f)$. Using matric multiplication we have

$$f = [x, y]\begin{bmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

or $f=(x, y)F(x, y)^T$.* We say that two forms f_1 and f_2 are *equivalent* if there is a unimodular transformation taking one into the other, that is, if f becomes $f'=a'x'^2+b'x'y'+c'y'^2$ in virtue of the transformation $(x, y)=(x', y')T$, T being unimodular. Two equivalent forms f_1 and f_2 are said to be in the same class and we write $f_1 \cong f_2$. If there is a unimodular transformation of determinant $+1$ taking f_1 into f_2 we say that the forms are properly equivalent and that they are in the same proper class; if there is a unimodular transformation of determinant -1 taking one form into the other the forms are called improperly equivalent. Two form f_1 and f_2 of the same determinant are said to be in the same genus if one can be taken into the other by a linear transformation with rational elements whose denominators are prime to twice the determinant of the forms; we then write $f_1 \text{ v } f_2$.

Suppose $\mathfrak{J}=[ua,\ ue+u\sigma]$. Then $N(uax+u(e+\sigma)y) = u^2N(ax+(e+\sigma)y) = u^2a(ax^2+bxy+cy^2)$ where

$$b = 2e + \sigma + \sigma^e, \qquad ac = N(e+\sigma). \tag{3}$$

Furthermore $ac-\frac{1}{4}b^2 = -\frac{1}{4}(\sigma-\sigma^c)^2 = -\frac{1}{4}\Delta$. Moreover, we have seen that any basis of $\mathfrak{J}$ may be obtained from its reduced basis by a unimodular transformation. This transformation will leave unaltered the norm and determinant of the ideal and takes f into an equivalent form. Thus we have proved

THEOREM 5. *If* $\mathfrak{J}=[\alpha_1, \alpha_2]$ *is in* $J(\Delta)$, *then*

$$N(\alpha_1x + \alpha_2y) = N(\mathfrak{J})f(x, y) \tag{4}$$

where $f(x, y)=ax^2+bxy+cy^2$ and $d(f)=-\frac{1}{4}\Delta$, *that is,* $d(\mathfrak{J})=N^2(\mathfrak{J})d(f)$. *If the basis is reduced, the relationship between the basis and the coefficients of* f *is given by equations* (3).

* The superscript T denotes "transpose" and I denotes "inverse."

The following theorem establishes the correspondence in the other direction.

THEOREM 6. *With a class of forms there is associated by* (5) *below an ideal* $\mathfrak{J}$ *in* $J(\Delta)$ *where* $b^2-4ac=\Delta$ *and for some basis of* $\mathfrak{J}$ *and some form* f *of the class, equation* (4) *holds with* $f=ax^2+bxy+cy^2$.

To prove this suppose that m is an integer least in absolute value represented by the class of forms. Then there must be a solution in integers x_0, y_0 of the equation $f=m$. Now x_0 and y_0 must be relatively prime since the square of any common factor would divide m denying the supposition that m is least. We may then determine integers v and w such that $x_0v-y_0w=1$ and the transformation

$$\begin{bmatrix} x_0 & w \\ y_0 & v \end{bmatrix}$$

is unimodular and will take f into a form whose leading coefficient is m. Hence we write f in the form in Theorem 6 with a a non-zero integer least in absolute value of all the numbers represented by f. Then with f we associate the ideal

$$I = \left(a, \frac{b+\sqrt{b^2-4ac}}{2}\right) = (\alpha_1, \alpha_2) \tag{5}$$

in $J(\Delta)$ where $\Delta=b^2-4ac$, and see that $N(\alpha_1x+\alpha_2y)=af$. It remains to show that $a=N(\mathfrak{J})$ and hence (α_1, α_2) is a basis. If $\Delta\equiv 1 \pmod 4$ we may take $r=a$, $s=(b-1)/2$, $u=1$ and, since $\sigma=\frac{1}{2}(1+\sqrt{b^2-4ac})=\frac{1}{2}(1+\sqrt{\Delta})$, we have $N(s+u\sigma)=ac\equiv 0 \pmod a$ and the conditions of Theorem 2 are satisfied. If $\Delta\not\equiv 1 \pmod 4$, b is even and we may take $s=\frac{1}{2}b$, $u=1$, $\sigma=\frac{1}{2}\sqrt{\Delta}$ and again $N(s+u\sigma)=ac\equiv 0 \pmod a$. Furthermore the correspondence between coefficients and basis is the same as in (3).

Notice that the form which we have made correspond to $\mathfrak{J}$ depends on the particular basis but that the class of that form is independent of the basis. In both cases the ideal $\mathfrak{J}=[\alpha_1, \alpha_2]$ and the form f satisfy condition (4). However, though all forms associated with any ideal by the above means are properly or improperly equivalent to one another, there may be several ideals associated with one form; for instance $\mathfrak{J}$ and $\rho\mathfrak{J}$, with ρ a quadratic integer, are associated with the same class of forms. To obtain uniqueness of correspondence we say that two ideals $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are in the same class (or ideal class) if there exist quadratic integers ρ_1 and ρ_2 such that $\rho_1\mathfrak{J}_1=\rho_2\mathfrak{J}_2$. Let Σ denote an ideal class (that is, the set of all ideals in a given class) and Γ the set of all forms (properly or improperly) equivalent to a given form. As above, Σ^c denotes the class of ideals obtained from Σ by replacing each number by its conjugate. We prove

THEOREM 7. *Two primitive ideals* $\mathfrak{J}_1$ *and* $\mathfrak{J}_2$ *in* $J(\Delta)$ *are associated by* (4) *with the same class* Γ *of forms if and only if the ideals are in the same class* Σ *or conjugate classes* Σ *and* Σ^c.

To prove this let $\mathfrak{J}_i=[\alpha_i, \beta_i]$ and $N(\alpha_ix_i+\beta_iy_i)=N(\mathfrak{J}_i)f_i(x_i, y_i)$ for $i=1, 2$

and $f_1(x_1, y_1) \cong f_2(x_2, y_2)$. If T is a unimodular transformation taking f_1 into f_2 we have the matric equation $(x_1, y_1) = (x_2, y_2)T$ and $f_1(x_1, y_1) = f_2(x_2 y_2)$ identically in x_2 and y_2. Now $(\alpha_1, \beta_1)^T = T^I(\alpha_1', \beta_1')^T$ defines a new basis α_1', β_1' of $\mathfrak{J}_1$ and $(x_1, y_1)(\alpha_1, \beta_1)^T = (x_2, y_2)(\alpha_1', \beta_1')^T$. Thus $N(\alpha_1 x_1 + \beta_1 y_1) = N(\mathfrak{J}_1) f_2(x_2, y_2)$ implies $N(\alpha_1' x_2 + \beta_1' y_2) = N(\mathfrak{J}_1) f_2(x_2, y_2)$ and replacing α_1', β_1' by α_1, β_1 we have

$$N(\alpha_i x_2 + \beta_i y_2) = N(\mathfrak{J}_i) f_2(x_2, y_2), \qquad i = 1, 2 \tag{6}$$

identically in x_2 and y_2. If a and c are the coefficients of x_2^2 and y_2^2 (they must be different from zero) respectively in $f_2(x_2, y_2)$ we have, taking the pairs of values (1, 0) and (0, 1)

$$N(\alpha_i) = N(\mathfrak{J}_i)a, \qquad N(\beta_i) = N(\mathfrak{J}_i)c, \tag{7}$$

and hence

$$N(\alpha_1)N(\mathfrak{J}_2) = N(\alpha_2)N(\mathfrak{J}_1), \qquad N(\beta_1)N(\mathfrak{J}_2) = N(\beta_2)N(\mathfrak{J}_1). \tag{8}$$

Then the ideals $\mathfrak{J}_4 = \alpha_1 \mathfrak{J}_2$ and $\mathfrak{J}_3 = \alpha_2 \mathfrak{J}_1$ have equal norms by (8) and Theorem 4 and are associated with the same class Γ of forms. Furthermore, $\mathfrak{J}_4$ and $\mathfrak{J}_3$ are in the same ideal class as $\mathfrak{J}_2$ and $\mathfrak{J}_1$ respectively. Thus, multiplying (6) by $N(\alpha_2)$ and $N(\alpha_1)$ respectively we have, equating coefficients of x_2^2, $x_2 y_2$, y_2^2 on the left side,

$$\alpha_4 \alpha_4^c = \alpha_3 \alpha_3^c, \qquad \alpha_4 \beta_4^c + \alpha_4 \beta_4^c = \alpha_3 \beta_3^c + \alpha_3^c \beta_3, \qquad \beta_4 \beta_4^c = \beta_3 \beta_3^c \tag{9}$$

where $\mathfrak{J}_4 = [\alpha_4, \beta_4]$ and $\mathfrak{J}_3 = [\alpha_3, \beta_3]$. But the determinants of $\mathfrak{J}_4$ and $\mathfrak{J}_3$ are equal since they depend only on Δ and the norms. Hence

$$\alpha_4 \beta_4^c - \alpha_4^c \beta_4 = \pm (\alpha_3 \beta_3^c - \alpha_3^c \beta_3) \tag{10}$$

If the positive sign holds we have, adding (10) to the second equation of (9) $\alpha_4 \beta_4^c = \alpha_3 \beta_3^c$ which, with $\beta_4 \beta_4^c = \beta_3 \beta_3^c$, implies $\alpha_4/\beta_4 = \alpha_3/\beta_3$ and hence $\mathfrak{J}_4$ and $\mathfrak{J}_3$ are in the same ideal class. If the negative sign holds, $\alpha_4 \beta_4^c = \alpha_3^c \beta_3$ which, with $\beta_4 \beta_4^c = \beta_3 \beta_3^c$, gives $\alpha_4/\beta_4 = \alpha_3^c/\beta_3^c$ and $\mathfrak{J}_4$ and $\mathfrak{J}_3^c$ are in the same class. This completes the proof that $f_1 \cong f_2$ implies that the corresponding ideal classes are equal or conjugate.

On the other hand, if $\Sigma_1 = \Sigma_2^c$ we may choose $[\alpha_1, \alpha_2]$ and $[\alpha_1^c, \alpha_2^c]$ as bases and $N(\alpha_1 x + \alpha_2 y) = = N(\alpha_1^c x + \alpha y_2^c)$ shows that the forms are equal.

Notice that if we define a form to be primitive when 1 is the g.c.d. of its coefficients, the definition of a primitive ideal and equations (3) show that a form is primitive if and only if the corresponding ideal class is primitive. Then our correspondence will be completed by the following result.

THEOREM 8. *The primitive ideal classes Σ and Σ^c are equal if and only if each corresponding class Γ is improperly equivalent to itself.*

To prove this we may take $[a, e+\sigma]$ as the representative of Σ and $[a, e+\sigma^c]$

as its conjugate. Suppose they are in the same class, that is, there exist quadratic integers α and β such that $\alpha\mathfrak{J}=\beta\mathfrak{J}^c$. Then $N(\alpha)\mathfrak{J}=\alpha^c\beta\mathfrak{J}^c$ and, since the coefficient of σ in every number of the ideal on the left is divisible by $N(\alpha)$ we see that $N(\alpha)$ divides $\alpha^c\beta$ and that $\mathfrak{J}=\gamma\mathfrak{J}^c$ for some quadratic integer γ. Taking the norm of both sides we see that $N(\gamma)$ is 1 and hence that the ideal (γ) is the ideal (1). Thus $\mathfrak{J}=\mathfrak{J}^c$ which implies that there exist integers t and v such that $ta+v(e+\sigma)=e+\sigma^c$. Equating coefficients of the rational and irrational parts we have $v=-1$ and $ta=2e+\sigma+\sigma^c$ which must be solvable for an integer t. Thus from (3) the form corresponding to the ideal is $ax^2+bxy+cy^2=f$ in which $b\equiv 0 \pmod{a}$. Then f may be taken into itself by the product of two transformations: first

$$H=\begin{bmatrix}1 & h\\ 0 & 1\end{bmatrix}$$

where h is chosen so that $2ah+b$ is 0 or a, and second

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}H^I \quad \text{or} \quad \begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}H^I$$

in the respective cases. Thus f is improperly equivalent to itself and the same will be true of every form in the class of f.

Conversely, if f may be taken into itself by the transformation T of determinant -1, we have $T^TFT=F$ where F is the matrix of f. Then for S any properly unimodular transformation, S^ITS takes S^TFS into itself where S^TFS is in the same proper class as F. We wish to choose S so that S^ITS has one of two simple forms. First we show that $T^TFT=F$ where T is the matrix with elements t_{ij} implies that $t_{11}+t_{22}=0$, that is, $\lambda_1+\lambda_2=0$ where λ_1 and λ_2 are the roots of $|T-\lambda I|=0$, the vertical lines denoting the determinant of the symbol enclosed. To this end notice that $\lambda_1\lambda_2=|T|=-1$ and let

$$R=\begin{bmatrix}-t_{12} & -t_{12}\\ t_{11}-\lambda_1 & t_{11}-\lambda_2\end{bmatrix}.$$

Then either $t_{12}=0$ in which case $|T|=-1$ implies $t_{11}t_{22}=-1$ and hence $t_{11}+t_{22}=0$ or else R is nonsingular since $\lambda_1\lambda_2=-1$ and $\lambda_1+\lambda_2$ real implies $\lambda_1\neq\lambda_2$. Then R^ITR has the form

$$\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$$

in view of the fact that $\lambda_i^2-(t_{11}+t_{22})\lambda_i-1=0$, and takes R^TFR into itself. Thus if $g=ax^2+bxy+cy^2$ is the form whose matrix is R^TFR (a, b, c need not be integral here) R^ITR takes it into $\lambda_1^2ax^2+b\lambda_1\lambda_2xy+c\lambda_2^2y_2^2$. Since $\lambda_1\lambda_2=-1$ we have $b=0$ and since the determinant of g is not zero we have $\lambda_1^2=\lambda_2^2=1$. Hence λ_1 and λ_2 are 1 and -1 in some order and their sum is zero. Take $\lambda_1=1$, $\lambda_2=-1$.

Next let u be the g.c.d. of t_{12} and $t_{11}-1$ and let $-t_{12}=us_{11}$ and $t_{11}-1=us_{21}$ define integers s_{11} and s_{21}. Since the latter are relatively prime we can determine s_{12} and s_{22} so that

$$S_1 = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}$$

has determinant 1. Then, since the first column of S_1 is proportional to that of R we have

$$TS_1 = S_1 \begin{bmatrix} 1 & r \\ 0 & s \end{bmatrix}$$

for integers r and s. Furthermore $|T| = -1$, $|S_1| = 1$ implies $s = -1$.

If we define the matrices

$$S_2 = \begin{bmatrix} 1 & (w-r)/2 \\ 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & w \\ 0 & -1 \end{bmatrix}$$

where $w=1$ or 0 according as r is odd or even, we see that S_2 is properly unimodular and

$$\begin{bmatrix} 1 & r \\ 0 & -1 \end{bmatrix} S_2 = S_2 D.$$

Then, letting $S=S_1S_2$ we have $TS=SD$ or $S^{\prime}TS=D$. Hence $F_1=S^TFS$ defines a matrix F_1 for which $T^TFT=F$ implies $D^TF_1D=F_1$.

Thus we have shown that if f can be taken into itself by a transformation of determinant -1, there is in the same proper class as f a form f_1 which is taken into itself by the transformation D above with $w=1$ or 0. Write $f_1=ax^2+bxy+cy^2$ and see that D takes f_1 into $a(x+wy)^2+b(x+wy)(-y)+cy^2=ax^2+xy(2aw-b)+(aw^2-bw+c)y^2$. Thus $b=2aw-b$ and $b\equiv 0 \pmod{a}$. Then, retracing our argument in the first part of this theorem we see that the corresponding ideal $\mathfrak{J}$ has the property that $\mathfrak{J}^c=\mathfrak{J}$. This completes the proof and permits us to make the definition: an ideal class Σ is called ambiguous if it is equal to its conjugate Σ^c. A class of quadratic forms is ambiguous if any (and hence every) form in the class may be taken into itself by a transformation of determinant -1. Hence we have just shown that there is a 1-1 correspondence between ambiguous classes of primitive ideals and ambiguous classes of primitive forms while the correspondence for non-ambiguous classes is 2-2. We therefore understand that in any correspondence between Σ and Γ classes, if Σ corresponds to Γ then Σ^c corresponds to Γ^c. It is this correspondence which enables us to prove expeditiously several important properties of binary quadratic forms.

4. Composition of ideal classes and classes of forms. If Γ_1 and Γ_2 are classes of forms associated with the ideal classes Σ_1 and Σ_2 in $J(\Delta)$ we define the class as-

sociated with $\Sigma_1\Sigma_2$ in $J(\Delta)$ to be the class $\Gamma_1\Gamma_2$ and this class is said to be derived from Γ_1 and Γ_2 by composition. Since Δ determines the determinant of the forms, their determinants are all equal. Furthermore, composition is commutative and associative since the product of ideals has these properties. If $\mathfrak{J}_i=[\alpha_i, \beta_i]$ we have, from equations (7) that $N(\alpha_i)=N(\mathfrak{J}_i)a_i$ where a_i is the leading coefficient of the form f_i associated with $\mathfrak{J}_i$. If the ideals $\mathfrak{J}_i$ are primitive, $N(\mathfrak{J}_1)N(\mathfrak{J}_2)=N(\mathfrak{J}_1\mathfrak{J}_2)$ implies $N(\alpha_1, \alpha_2)=N(\mathfrak{J}_1\mathfrak{J}_2)a_1a_2$. Thus a_1a_2 will be the leading coefficient of some form in $\Gamma_1\Gamma_2$. Hence we have proved the following important theorem:

THEOREM 9. *If a_1 and a_2 are represented by primitive forms of classes Γ_1 and Γ_2 then a_1a_2 is represented by forms of the class $\Gamma_1\Gamma_2$.*

We shall need the following two results on binary quadratic forms.

THEOREM 10. *A form f is primitive if and only if it represents a number prime to $8d(f)$.*

To prove this let p_i be any prime factor of $8d(f)$ where $f=ax^2+bxy+cy^2$. If p_i is prime to a, let $x_i=1$, $y_i=0$; if p_i is a divisor of a and not of c let $x_i=0$, $y_i=1$; if p_i divides both a and c, it does not divide b and we let $x_i=y_i=1$; for these cases the value of f will be prime to p_i. By the Chinese remainder theorem we can choose $x\equiv x_i \pmod{p_i}$, $y\equiv y_i \pmod{p_i}$ for all prime divisors p_i of $8d(f)$ and, for such an x and y, f will be prime to $8d(f)$. On the other hand, if g is the g.c.d. of the coefficients of f, it is a divisor of $8d(f)$ and divides all numbers represented by f.

THEOREM 11. *Two primitive binary quadratic forms f and g with integral coefficients are in the same genus if and only if their determinants are equal and there are integers a and w prime to $8d(f)$ such that $f=a$ and $g=w^2a$ are solvable in integers.*

We know by Theorem 10 that f represents a number a prime to $8d(f)$ and hence we may take f to be $ax^2+bxy+cy^2$. If f and g are of the same genus we know from the definition of genus that there is a transformation

$$\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} w^{-1}$$

taking g into f where the t_{ij} are integers and w is an integer prime to $8d(f)$. Then $(t_{11}/w, t_{21}/w)$ is a solution of $g=a$ and hence $g=aw^2$ is solvable in integers.

Conversely suppose $g=w^2a$ has a solution t_{11}, t_{21} in integers. Then there are integers u and v such that $t_{11}v-t_{21}u=q$ where q is the g.c.d. of t_{11} and t_{21} and hence q^2 divides w^2a. Then the transformation

$$\begin{bmatrix} t_{11}/w & uw/q \\ t_{21}/w & vw/q \end{bmatrix} = T$$

has determinant 1 and takes g into $g'=ax^2+b'xy+c'y^2$ where b' and c' need not be integers while $qb'\equiv b \pmod 2$. Since q^2 divides w^2a, the denominators of the elements of T and of b' and c' are prime to $8d(f)$. Then the transformation

$$S=\begin{bmatrix}1 & (b-b')/2a\\ 0 & 1\end{bmatrix}$$

takes g' into $g''=ax^2+bxy+c''y^2$. Since $d(g'')=d(g)=d(f)$ we have $c''=c$ and $g''=f$. Hence TS takes g into f and TS is a transformation with rational elements whose denominators are prime to $8d(f)$. This shows that f and g are in the same genus.

Since the genus of a form is determined by any odd integer prime to the determinant and represented by the form we have

THEOREM 12. *If* $\Gamma_1 \vee \Gamma_1'$ *and* $\Gamma_2 \vee \Gamma_2'$, *then* $\Gamma_1\Gamma_2 \vee \Gamma_1'\Gamma_2'$.

If a form represents 1 we say that it is in the principal class Γ_0 and call the corresponding ideal class the principal ideal class Σ_0. The genus containing the principal class is called the principal genus. Notice that the principal ideal class contains the ideal $(1)=J$. We now prove

THEOREM 13. *The primitive ideal classes of a given* $J(\Delta)$ *and hence the primitive classes of forms, form a multiplicative group. Also* $\Sigma\Sigma^c=\Sigma_0$ *and* $\Gamma\Gamma^c=\Gamma_0$ *for primitive classes* Σ *and* Γ *and their conjugates.*

First we see that Γ_1 and Γ_2 primitive imply that $\Gamma_1\Gamma_2$ is primitive since Theorem 10 shows that forms f_1 and f_2 of Γ_1 and Γ_2, respectively, represent numbers a_1 and a_2 prime to $8d(f)$, and hence, by Theorem 9 forms of $\Gamma_1\Gamma_2$ represent a number a_1a_2 prime to $8d(f)$ which shows by Theorem 10 that $\Gamma_1\Gamma_2$ is primitive.

Now Lemma 1 shows that if $\mathfrak{J}$ is a primitive ideal. $\mathfrak{J}\cdot\mathfrak{J}^c$ is the principal ideal $(N(\mathfrak{J}))=N(\mathfrak{J})\cdot(1)$ and hence $\mathfrak{J}\cdot\mathfrak{J}^c$ is in the principal ideal class.

The principal ideal class is the unit element of the group. We have shown the existence of the inverse and the closure property. Furthermore the associative property holds from the same property for quadratic integers. Hence we have shown that the primitive ideal classes form a group and, if we choose our correspondence so that $\Gamma_0\Gamma=\Gamma$ for every Γ we have the group properties for the classes of primitive forms.

5. Consequences of the composition of forms. We first prove the rather startling result embodied in

THEOREM 14. *If* h *is the number of proper classes of forms in the principal genus for a given determinant, then the number of proper classes in each genus of primitive forms is* h.

To prove this let $\Gamma_0, \Gamma_2, \cdots, \Gamma_h$ be the classes in the principal genus and let Γ be a representative of any other genus. Then $\Gamma\Gamma_0, \cdots, \Gamma\Gamma_h$ are all in the

genus of Γ by Theorem 12. No two are in the same class since $\Gamma\Gamma_i=\Gamma\Gamma_j$ would imply, by multiplication by Γ^c, that $\Gamma_i=\Gamma_j$.

Next we have Gauss's celebrated theorem on duplication.

THEOREM 15. *If Γ is any class in the principal genus of primitive forms there is a class Γ_1 such that $\Gamma_1^2=\Gamma$.*

Let $\mathfrak{J}=[a, e'+\sigma]$ be an ideal corresponding to Γ. Then $N\{ax+(e'+\sigma)y\}=N(\mathfrak{J})f(x, y)=af(x, y)$ where $f(x, y)=ax^2+bxy+cy^2$. If g is a form in the principal class it represents 1 and f v g implies the existence of a transformation

$$\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} w^{-1}$$

of determinant 1 taking f into g with w prime to $8d(f)$ and t_{ij} integers. That is, $f=1$ has a solution $(t_{11}/w, t_{12}/w)$. Hence $f=w^2$ has a solution t_{11}, t_{21} in integers. Since any common factor of t_{11} and t_{21} divides w and may be removed, we may consider t_{11} and t_{21} to be relatively prime. Then t'_{12} and t'_{22} may be chosen so that

$$\begin{bmatrix} t_{11} & t'_{12} \\ t_{21} & t'_{22} \end{bmatrix}$$

is unimodular and takes f into $w^2x^2+b'xy+c'y^2$ where b' and c' are integers. Hence we may assume w^2 to be the leading coefficient of f and an ideal associated with f is $\mathfrak{J}=[w^2, e+\sigma]$.

Our proof will be complete if we can show that $I_1^2=\mathfrak{J}$ where $\mathfrak{J}_1=[w, e+\sigma]$. Since w is prime to 2Δ, it is also prime to $b'=2e+\sigma+\sigma^e$ and hence there are integers x and y such that $wx+(2e+\sigma+\sigma^e)y=1$. Now $\mathfrak{J}_1^2=(w^2, w(e+\sigma), (e+\sigma)^2)$ and hence $\mathfrak{J}_1^2$ contains $xw(e+\sigma)+y(e^2+2e\sigma+\sigma^2)=e+\sigma-yN(e+\sigma)$. But $N(e+\sigma)\equiv 0 \pmod{w^2}$ from Theorem 2 and hence $e+\sigma$ is in $\mathfrak{J}_1^2$ which shows that $\mathfrak{J}_1^2=[w^2, e+\sigma]$ and completes the proof.

Finally we prove

THEOREM 16. *The number of primitive ambiguous classes of given determinant is equal to the number of genera of primitive forms.*

Since a class Γ is ambiguous if and only if $\Gamma^2=\Gamma_0$ we see that the ambiguous classes form a subgroup of the group of classes of all primitive forms of the given determinant. Also $\Gamma_1^2=\Gamma_2^2$ implies $\Gamma_1=\Gamma_2\Gamma_a$ where Γ_a is ambiguous. Let q be the number of ambiguous classes, g the number of genera and h the number of classes in each genus. Then gh/q is the number of distinct squares of classes. This, from Theorem 15, is equal to h and hence $g=q$.

Notice that if there are q' ambiguous classes in the principal genus, each genus has q' or 0 ambiguous classes.

MATHEMATICAL NOTES

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AN OPERATOR APPROACH TO MATRIX THEOREMS

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1. Introduction. A known theorem on matrices is the following:[1]

If a matrix has two of the three properties in one of the following sets, then it has all three properties in that set:

(a) *Real, orthogonal, unitary*;
(b) *Symmetric, orthogonal, involutory*;
(c) *Hermitian, unitary, involutory.*

A generalization of the above theorem will be given whose proof employs linear operators rather than the matrices themselves. Finally, a finite projective geometry will be exhibited to schematize the algebraic results. This geometric representation unifies certain matrix concepts which heretofore have been treated only separately, if at all.

2. Notation. Matrix A will be called regular provided it is complex and nonsingular. Hereafter all matrices considered will be regular, and no mention of their elements will be made.

Define eA, kA, nA, tA, vA to be the matrices A, $\overline{A}$, $-A$, A', A^{-1}, respectively, of our customary notation. Let (A, B) denote a matrix couple, i.e., an ordered pair of matrices. Define $p(A, B)$, $q(A, B)$, and $r(A, B)$ to be the matrix AB, the matrix couple (nA, B), and the matrix couple (B, A), respectively.

Let the domain of an operator be the set of all elements for which the operator is defined, and let the range of an operator be the set of all image elements. Denote the operators e, k, n, t, v by x_i, and the operators e, q, r by z_i. Thus the domain of operator x_i is a set of matrices, while the domains of operators p, q, and r are sets of matrix couples. We now extend the domain of x_i to matrix couples by defining:

$$x_i(A, B) =_{Df} (x_iA, x_iB).$$

Note, however, that the range of operator p is a set of matrices, while the ranges of operators q and r are sets of matrix couples.

Equality of two operators is designated by $\doteq$. The product of two operators is designated by ordered juxtaposition beginning at the right.

3. Operator Theorems. Using the preceding definitions and notation we state several operator theorems without proof. These operator theorems can be

[1] See C. C. MacDuffee, The Theory of Matrices, p. 25, Theorem 18.1.

translated into their corresponding matrix theorems simply by inspection.

INVOLUTION THEOREM: $(Vx_i)[x_i^2 \doteq e]$.
COMMUTATION THEOREM: $(Vx_i)(Vx_j)[x_ix_j \doteq x_jx_i]$.
INVOLUTION THEOREM: $(Vz_i)[z_i^2 \doteq e]$.
ANTI-COMMUTATION THEOREM: $rq \doteq qrn$.
COMMUTATION THEOREM: $(Vx_i)(Vz_j)[x_iz_j \doteq z_jx_i]$.
EXISTENCE THEOREM: $(Vx_i)(\exists z_i)[x_ip \doteq px_iz_i]$.

4. Standard Subsets and Properties. We now define 14 proper subsets of the group of regular matrices. We call them standard subsets, and they in turn define 14 matrix properties, called standard properties, each of which corresponds to its particular subset as follows:

Standard Property	*Customary Notation*	*Operator Notation*	*Operator Equation*	*Abbreviation of Standard Property*
1. A is real	$A = \overline{A}$	$eA = kA$	$e \doteq k$	R
2. A is pure imaginary	$A = -\overline{A}$	$eA = nkA$	$e \doteq nk$	
3. A is symmetric	$A = A'$	$eA = tA$	$e \doteq t$	S
4. A is skew symmetric	$A = -A'$	$eA = ntA$	$e \doteq nt$	
5. A is involutory	$A = A^{-1}$	$eA = vA$	$e \doteq v$	I
6. A is skew involutory	$A = -A^{-1}$	$eA = nvA$	$e \doteq nv$	
7. A is orthogonal	$A = A'^{-1}$	$eA = vtA$	$e \doteq vt$	O
8. A is skew orthogonal	$A = -A'^{-1}$	$eA = nvtA$	$e \doteq nvt$	
9. A is Hermitian	$A = \overline{A}'$	$eA = ktA$	$e \doteq kt$	H
10. A is skew Hermitian	$A = -\overline{A}'$	$eA = nktA$	$e \doteq nkt$	
11. A is nominal[2]	$A = \overline{A}^{-1}$	$eA = vkA$	$e \doteq vk$	N
12. A is skew nominal	$A = -\overline{A}^{-1}$	$eA = nvkA$	$e \doteq nvk$	
13. A is unitary	$A = \overline{A}'^{-1}$	$eA = vktA$	$e \doteq vkt$	U
14. A is skew unitary	$A = -\overline{A}'^{-1}$	$eA = nvktA$	$e \doteq nvkt$	

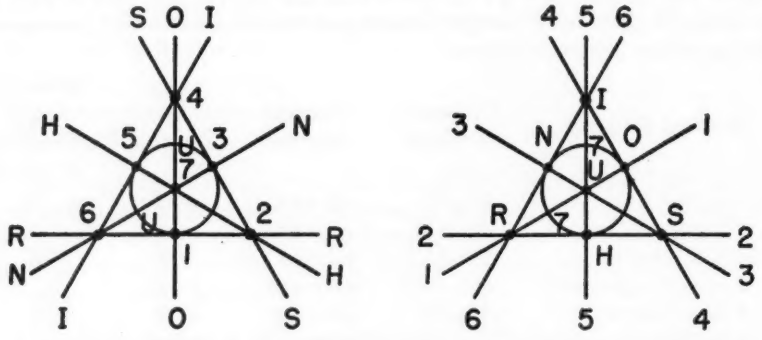
5. Theorem. Let a set closed, associative, and commutative under a binary operation be called an abelian semigroup. We wish to find those sets of standard properties which form semigroups under the operation of property conjunction. Let $x_i \doteq e$, k, t, v. Since $x_1 \doteq e$ and $x_2 \doteq e$ imply $x_1x_2 \doteq e$, examination of the next to the last column in the above table reveals seven closed sets of standard properties. Further, since the operation of property conjunction is associative and commutative, we have established the following theorem:

THEOREM: *Each of the sets* 1: $\{R, U, O\}$, 2: $\{R, S, H\}$, 3: $\{S, N, U\}$, 4: $\{S, O, I\}$, 5: $\{H, I, U\}$, 6: $\{R, N, I\}$, 7: $\{O, N, H\}$ *is an abelian semigroup under property conjunction.*

[2] This term is introduced by the author for an un-named type mentioned briefly by Turnbull and Aitken, An Introduction to the Theory of Canonical Matrices, p. 33.

6. Projective Geometric Representation. We present the finite projective geometry $PG(2, 2)$ and its dual to geometrize the algebraic results. In the original geometry a line denotes a matrix property, a point denotes a semigroup, the incidence of a point on a line denotes that a semigroup contains a property, and the intersection of two lines denotes the conjunction of two properties. Here a necessary and sufficient condition that three properties form a semigroup is that they lie on a point. In the dual a necessary and sufficient condition that three properties form a semigroup is that they lie on a line.

Projective Geometry $PG(2, 2)$ Dual



In conclusion it should be mentioned that the Involution, Commutation, and Existence Theorems for our operators permit a novel proof of the proposition that of all standard subsets only the real, orthogonal, and unitary ones form groups under matrix multiplication. This proof, though direct, is too long to be given here.

AN ELEMENTARY PROOF OF STÄCKEL'S THEOREM

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The system associated in the classical dynamics with the name of Stäckel is a holonomic system with n degrees of freedom, for which the kinetic energy function T and the potential energy function V have the particular forms

$$T = \frac{1}{2}\sum_r \frac{1}{v_{1r}}\dot{q}_r^2, \qquad V = \sum_r v_{1r}w_r,$$

where the summations run from 1 to n, and the functions v_{1r} and w_r which occur in the formulae have the following properties:

i. There exists a matrix (u_{rs}) in which the elements u_{rs} of the rth row are functions only of the rth Lagrangian coordinate q_r; the motion takes place in a domain D of the q-space in which the determinant U of (u_{rs}) does not vanish; (v_{rs}) is the matrix inverse to (u_{rs}). (In particular the elements v_{1r} which occur in

the formula for T are the minors of the elements in the first *column* of the matrix (u_{rs}) divided by U.)

ii. The coefficient w_r is a function only of q_r.

iii. The element u_{rs} has a continuous derivative with respect to q_r, and the coefficient w_r has a continuous derivative with respect to q_r, in D. (It follows that the elements v_{rs} have continuous first derivatives with respect to all the q's.)

Stäckel's theorem asserts that there are n first integrals of the equations of motion, namely

$$\sum_r v_{kr}\left(\frac{1}{2}\frac{\dot{q}_r^2}{v_{1r}^2} + w_r\right) = \alpha_k, \qquad k = 1, 2, \cdots, n,$$

where the α_k are constants. (The special case $k=1$ is of course the familiar integral of energy.)

The natural context for the theorem is the general dynamical theory of Hamilton and Jacobi, and the proof usually given (for example in Stäckel's papers [1] and the standard text-books [2]) presents the result as a corollary of the Hamilton-Jacobi theorem. It is not difficult however to derive the result directly from the Lagrangian equations of motion.

We prove first the following Lemma: *For all values of s, k, and r we have*

$$v_{1s}\frac{\partial v_{kr}}{\partial q_s} - v_{ks}\frac{\partial v_{1r}}{\partial q_s} = 0.$$

The proof of this Lemma is simple. We have, for all values of k and m,

$$\sum_r v_{kr}u_{rm} = \delta_{km},$$

whence, differentiating with respect to q_s, and remembering that u_{rm} is a function of q_r only, we obtain

$$\sum_r \frac{\partial v_{kr}}{\partial q_s}u_{rm} + v_{ks}\frac{du_{sm}}{dq_s} = 0.$$

If we write the same equation with 1 in place of k, and eliminate du_{sm}/dq_s, we find

$$\sum_r u_{rm}\left(v_{1s}\frac{\partial v_{kr}}{\partial q_s} - v_{ks}\frac{\partial v_{1r}}{\partial q_s}\right) = 0.$$

If we denote the expression in the parentheses by θ_r (k and s being fixed) we have n equations

$$\sum_r u_{rm}\theta_r = 0,$$

one for each value of m. The determinant of the coefficients is U, and does not

vanish, so each $\theta_r = 0$. This proves the Lemma.

To deduce Stäckel's theorem we have

$$\frac{d}{dt}\sum_r v_{kr}\left(\frac{1}{2}\frac{\dot{q}_r^2}{v_{1r}^2}+w_r\right) = \sum_r\left(\frac{1}{2}\frac{\dot{q}_r^2}{v_{1r}^2}+w_r\right)\frac{d}{dt}v_{kr}+\sum_r v_{kr}\left(\frac{\dot{q}_r}{v_{1r}}\right)\frac{d}{dt}\left(\frac{\dot{q}_r}{v_{1r}}\right)+\sum_r v_{kr}\frac{dw_r}{dt}.$$

Substituting for $d/dt(\dot{q}_r/v_{1r})$ from Lagrange's equations of motion, we obtain

$$\begin{aligned}&\sum_r\left(\frac{1}{2}\frac{\dot{q}_r^2}{v_{1r}^2}+w_r\right)\sum_s\frac{\partial v_{kr}}{\partial q_s}\dot{q}_s+\sum_r v_{kr}\frac{\dot{q}_r}{v_{1r}}\left\{-\frac{1}{2}\sum_s\frac{\dot{q}_s^2}{v_{1s}^2}\frac{\partial v_{1s}}{\partial q_r}\right.\\&\qquad\left.-\frac{\partial}{\partial q_r}\left(\sum_s v_{1s}w_s\right)\right\}+\sum_r v_{kr}\frac{dw_r}{dq_r}\dot{q}_r\\&=\sum_{r,s}\frac{1}{2}\frac{\dot{q}_r^2\dot{q}_s}{v_{1r}^2}\frac{\partial v_{kr}}{\partial q_s}+\sum_{r,s}w_r\frac{\partial v_{kr}}{\partial q_s}\dot{q}_s-\sum_{r,s}\frac{1}{2}v_{kr}\frac{\dot{q}_r}{v_{1r}}\frac{\dot{q}_s^2}{v_{1s}^2}\frac{\partial v_{1s}}{\partial q_r}\\&\qquad-\sum_{r,s}v_{kr}\frac{\dot{q}_r}{v_{1r}}\frac{\partial v_{1s}}{\partial q_r}w_s-\sum_r v_{kr}\frac{\dot{q}_r}{v_{1r}}v_{1r}\frac{dw_r}{dq_r}+\sum_r v_{kr}\frac{dw_r}{dq_r}\dot{q}_r.\end{aligned}$$

The last two sums disappear, and, interchanging the dummy suffixes r and s in the third and fourth sums, we obtain

$$\begin{aligned}&\sum_{r,s}\frac{1}{2}\frac{\dot{q}_r^2\dot{q}_s}{v_{1r}^2}\frac{\partial v_{kr}}{\partial q_s}-\sum_{r,s}\frac{1}{2}\frac{\dot{q}_r^2\dot{q}_s}{v_{1r}^2}\frac{v_{ks}}{v_{1s}}\frac{\partial v_{1r}}{\partial q_s}+\sum_{r,s}w_r\frac{\partial v_{kr}}{\partial q_s}\dot{q}_s-\sum_{r,s}w_r\frac{v_{ks}}{v_{1s}}\frac{\partial v_{1r}}{\partial q_s}\dot{q}_s\\&\qquad=\sum_{r,s}\left(\frac{1}{2}\frac{\dot{q}_r^2}{v_{1r}^2}+w_r\right)\dot{q}_s\left(\frac{\partial v_{kr}}{\partial q_s}-\frac{v_{ks}}{v_{1s}}\frac{\partial v_{1r}}{\partial q_s}\right),\end{aligned}$$

which vanishes in virtue of the Lemma. The theorem follows.

The result has been known for more than half a century, so from the point of view of dynamical theory the present exposition is unimportant. From the point of view of the teaching of dynamics, however, it may be important; by means of it the theorem can be introduced and used in an elementary course which contains Lagrange's equations but does not embrace the more recondite theory of Hamilton and Jacobi.

References

1 Stäckel. Math. Ann., vol. 42, 1893, p. 537; Comptes Rendus, Paris, vol. 116, 1893, p. 485 and p. 1284, vol. 121, 1895, p. 489.

2. See for example Appel, *Mécanique Rationelle* (1923), vol. II, p. 433; Levi-Civitá e Amaldi, *Meccanica Razionale* (1927), vol. II, part 2, p. 420.

A THEOREM ON POSITIVE DEFINITE MATRICES

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Let us write

$$|A| = |A|_+ - |A|_-$$

for the complete expansion of the determinant of a matrix A, so that the first member on the right-hand side of the equation is the sum of those terms which have positive signs in the determinant expansion. Then we have the following result.

THEOREM: *If A is a positive definite Hermitian matrix then* $|A|_- \geqq 0$.

Proof. Let A be a positive definite Hermitian matrix. By a well known theorem* there exists an $n \times n$ triangular matrix (t_{ij}) with $t_{ij}=0$ for $j>i$ such that

$$A = (t_{ij})(\bar{t}_{ij})'. \tag{1}$$

By definition,

$$|A|_- = \sum a_{1i_1} \cdots a_{ni_n}, \tag{2}$$

the sum taken over all odd permutations of $1, \cdots, n$.

Using the decomposition (1) we have

$$a_{ik} = \sum_{r=1}^{n} t_{ir}\bar{t}_{kr}$$

and so

$$a_{1i_1} \cdots a_{ni_n} = \sum_{r_1=1}^{n} \cdots \sum_{r_n=1}^{n} t_{i_1r_1} \cdots t_{i_nr_n} \cdot \bar{t}_{1r_1} \cdots \bar{t}_{nr_n}.$$

Hence by (2)

$$|A|_- = \sum_{r_1=1}^{n} \cdots \sum_{r_n=1}^{n} |t_{ir_j}|_- \cdot \bar{t}_{1r_1} \cdots \bar{t}_{nr_n}. \tag{3}$$

The multiple sum in (3) has a value independent of the order of summation and this value is therefore the average of the sum corresponding to any set of permutations of the orders of summation. Take this set to be that of the R even permutations and observe that in each of them the first (determinantal) factor has the same value. The residual factors add up to $|\bar{t}_{ir_j}|_+$, and so we have

$$|A|_- = \sum_{r_1=1}^{n} \cdots \sum_{r_n=1}^{n} \frac{1}{R} |t_{ir_j}|_- \cdot |\bar{t}_{ir_j}|_+. \tag{4}$$

If $r_i=r_j$ for some $i \neq j$, then $|t_{ir_j}|=0$, and hence $|t_{ir_j}|_+ = |t_{ir_j}|_-$. Therefore

* See e.g. F. D. Murnaghan, The theory of group representations, Baltimore, 1938, pp. 20–22.

the two determinantal factors in (4) are conjugate and their product is positive. If no such i and j exist it will be shown that the product vanishes. For, in this case $|t_{iv_j}|$ has as its columns a permutation of the columns of $|t_{ij}|$ and the latter has in its formal expansion only one term different from 0. Hence either $|t_{iv_j}|_+$ or $|t_{iv_j}|_-$ will vanish. This establishes the theorem.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College and Institute for Advanced Study

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CERTAIN BASIC THEOREMS ON LINEAR DIFFERENTIAL EQUATIONS*

H. B. CURRY, Pennsylvania State College

1. Introduction. The theorems to be discussed are three, viz.: 1) the theorem that if n linearly independent solutions of a linear differential equation of the nth order are known, any other solution is a linear combination of them; 2) the theorem that the different solutions usually given for an equation with constant coefficients actually are linearly independent; and 3) the theorem stating the method of finding a particular integral of the nonhomogeneous equation with constant coefficients. These theorems can hardly be unfamiliar to you. Nevertheless, a survey of otherwise suitable textbooks (for an elementary course at The Pennsylvania State College) shows that in practically all of them the treatment of the first two theorems is logically inadequate, while that of the third theorem is unnecessarily complicated. Since the objective of this Association is the improvement of college teaching, it is appropriate to emphasize the presentation of these theorems here, even though that involves some reiteration of what has been known for a long time.

In regard to the first of these theorems, what is usually done is to show, quite correctly, that any linear combination of solutions (of the homogeneous equation) is a solution, and then to say that a linear combination, with arbitrary constant coefficients, of n solutions is "the general solution" because it contains n arbitrary constants. The only proof offered of the principle implicit here—viz., that a solution of an nth order equation embraces every solution if it depends on n arbitrary constants—is a heuristic argument in the front of the book. All this argument shows is that a general solution of the form stated is what one would normally expect; and the existence of singular solutions for $n=1$ shows that there are exceptions. An exact formulation and proof of the principle would require consideration of existence and uniqueness theorems for differential

* An address presented to the Allegheny Mountain Section of the Association at Pittsburgh, Pa., on November 6, 1948.

systems which are altogether too difficult for the kind of course here considered. Now it happens that the student solves equations of the first order by deriving necessary consequences of the equation, as in elementary algebra; and accordingly, if he has learned to reason correctly, he proves, in each particular case, that the solution or solutions which he finds are exhaustive. The principle mentioned is, therefore, not logically necessary for his argument; and so a heuristic and vague treatment of it is quite suitable. But in the study of linear equations one of the methods we wish to use gives sufficient, rather than necessary, conditions. In that case the first of our three theorems becomes an essential part of the logic. If we want to teach our students to reason correctly, an adequate proof of the theorem should be given. Somewhat similar remarks apply, of course, to the second theorem.

The proofs here given of these theorems are, in one sense, elementary in character. Of course, they are presented to you in a more abstract and general form than may be suitable for a class. The first two proofs, which depend on a general n, are by mathematical induction. However the proofs are intrinsically no more difficult than those of other theorems depending on general n with which the student is familiar, and they can be clarified by the usual devices of good teaching. No originality is claimed for the proofs. Parts of them will even be found in some existing textbooks, e.g. that of Agnew (cited below); but I know of no textbook in English which contains all of them.

2. The completeness theorem. Let $y_1, y_2, \cdots, y_n$ be n linearly independent solutions of the linear differential equation of the nth order

$$L(y) = 0. \tag{1}$$

Let Y be any other solution. Then it is to be shown that there exist constants $C_1, C_2, \cdots, C_n$ such that

$$Y = C_1y_1 + C_2y_2 + \cdots + C_ny_n. \tag{2}$$

This theorem is clear if $n=1$; it is, in fact, shown in the discussion of the linear equation of the first order. In order to establish it generally we employ mathematical induction. Assume the theorem true for equations of order $n-1$; we then show it is true for those of order n.

To this end we use the standard technique for reducing the order when one solution is known (and thus obtain that theorem as a by-product). If we substitute $Y=Zu$ in (1) we have an identity of the form

$$L(Zu) = ZL(u) + Z'L_1(u) + Z''L_2(u) + \cdots + Z^{(n)}L_n(u),$$

where the L_i are linear differential operators whose nature does not concern us. If u is a particular solution of (1), the term in Z vanishes and we have

$$L(uZ) = M(Z'), \tag{3}$$

where M is a linear differential operator of order $n-1$.

Now suppose $y_1, y_2, \cdots, y_n$ are linearly independent solutions of (1), and let Y be any other solution. Let

$$z_1 = \frac{y_1}{y_n}, z_2 = \frac{y_2}{y_n}, \cdots, z_{n-1} = \frac{y_{n-1}}{y_n}, Z = \frac{Y}{y_n}. \tag{4}$$

Then $z_1', z_2', \cdots, z_{n-1}', Z'$ are solutions v of

$$M(v) = 0,$$

as we see from (3) with $u = y_n$. Moreover $z_1', \cdots, z'_{n-1}$ are linearly independent. For if

$$C_1z_1' + C_2z_2' + \cdots + C_{n-1}z_{n-1}' = 0,$$

then, integrating, we have, for same constant C_n,

$$C_1z_1 + C_2z_2 + \cdots + C_{n-1}z_{n-1} + C_n = 0;$$

thence, multiplying by y_n and using (4), we have

$$C_1y_1 + C_2y_2 + \cdots + C_{n-1}y_{n-1} + C_ny_n = 0$$

and so all the C's are 0. By the hypothesis of the induction there must exist $C_1, C_2, \cdots, C_{n-1}$ such that

$$Z' = C_1z_1' + C_2z_2' + \cdots + C_{n-1}z_{n-1}'.$$

Hence, for some C_n,

$$Z = C_1z_1 + C_2z_2 + \cdots + C_{n-1}z_{n-1} + C_n.$$

Multiplying by y_n we have (2).

It will be noted that this argument involves nothing more difficult than the theorem on the reduction of the order for a linear differential equation—a theorem which is given in most textbooks.

3. Linear independence of the basis. The second theorem, on the linear independence of the basic solutions of an equation with constant coefficients is evidently equivalent to the following: If $p_1, p_2, \cdots, p_n$ are all distinct numbers, and $P_1, P_2, \cdots, P_n$ are polynomials, then

$$P_1e^{p_1x} + P_2e^{p_2x} + \cdots + P_ne^{p_nx} \tag{5}$$

cannot vanish identically unless all the P_i vanish identically.

This theorem is clear when $n = 1$. Suppose it true for a given n; we prove it is true for $n+1$.

Let $p_1, p_2, \cdots, p_n, p_{n+1}$ be distinct numbers and let $P_1, P_2, \cdots, P_{n+1}$ be polynomials such that

$$P_1e^{p_1x} + P_2e^{p_2x} + \cdots + P_ne^{p_nx} + P_{n+1}e^{p_{n+1}x} = 0. \tag{6}$$

If $P_{n+1} \equiv 0$, then all the $P_i = 0$ by the hypothesis of the induction. If not we can divide (6) by $P_{n+1}e^{p_{n+1}x}$, obtaining

$$R_1 e^{r_1 x} + R_2 e^{r_2 x} + \cdots + R_n e^{r_n x} + 1 = 0$$

where the R_i are rational, $r_i = p_i - p_{n+1}$. Differentiating, we have

$$(R_1' + r_1 R_1)e^{r_1 x} + \cdots + (R_n' + r_n R_n)e^{r_n x} = 0$$

If we clear of fractions, the left side becomes an expression of form (5) which vanishes identically. Since the common denominator does not vanish identically, the hypothesis of the induction shows that for all i

$$R_i' + r_i R_i = 0,$$
$$R_i = C_i e^{-r_i x}.$$

This contradicts the fact that the R_i are rational. Hence all P_i in (6) are identically 0.

4. Method of undetermined coefficients. As already stated the difficulty with the third theorem is that the general case is long and complicated—one author takes a full half page merely to state the theorem; furthermore the formulations usually given do not show why the method applies.

Suppose, then, we have to solve an equation

$$\phi(D)[y] = f(x). \tag{7}$$

It is supposed that we know how to find the complementary function, and particular solution of (7) is sought. The cases where the method of undetermined coefficients applies are those in which there exists an operator $\psi(D)$ such that

$$\psi(D)[f(x)] = 0. \tag{8}$$

$$\psi(D)\phi(D)[y] = 0. \tag{9}$$

In other words every solution y of (7) will be a solution of (9), and hence must be a specialization of the general solution of (9). Thus to get the general solution of (7) we can substitute the general solution of (9) in (7) and see what relations must hold among the arbitrary constants in order that (7) hold.

Furthermore, given any solution of (9) which is also a solution of (7), if we delete the terms which are part of the complementary function of (7), the remaining terms must also constitute a solution of (7). Hence a particular integral of (7) can be obtained by the following rule. Find $\psi(D)$ such that (8) holds; write the general solution of (9) and strike out terms in the complementary function of (7); then use the method of undetermined coefficients to specialize the constants so that (7) holds.

This rule entails the ones usually stated. Suppose, for instance,

$$f(x) = x^p e^{kx}.$$

Then

$$\psi(D) = (D - k)^{p+1}.$$

Let $\phi(D) = (D-k)^q\omega(D)$, where $\omega(k) \neq 0$. Then the general solution of (9) consists of

$$(C_0 + C_1x + C_2x^2 + \cdots + C_{p+q}x^{p+q})e^{kx} + \cdots$$

where the dots on the right indicate terms also present in the *c.f.* of (7). The first q terms indicated are also part of the *c.f.* of (7). Hence the particular integral is of the form

$$y = x^q(C_q + C_{q+1}x + \cdots + C_{p+q}x^p)e^{kx},$$

where the constants are to be determined by method of undetermined coefficients.†

PYTHAGOREAN TRIPLES

W. R. TALBOT, Jefferson City, Missouri

Pythagorean triples are such popular topics with undergraduates, a brief and simple method for their determination as special cases of the solution of the more general equation $x^2+y^2=z^n$ may be of interest.

The sum of two real squares as a^2+b^2 may be written as the square of the absolute value of the complex number $a+ib=\rho(\cos\theta+i\sin\theta)$. By DeMoivre's Theorem $(a+ib)^n=\rho^n(\cos n\theta+i\sin n\theta)$. Then $|(a+ib)^n| = \rho^n = |a+ib|^n$. Let $(a+ib)^n$ be expanded to give $A+iB$ where A and B are real. Then $|(a+ib)^n| = (A^2+B^2)^{1/2} = \rho^n = |a+ib|^n = (a^2+b^2)^{n/2}$. Then $A^2+B^2=(a^2+b^2)^n$. The solution of the equation is $x=A$, $y=B$, and $z=a^2+b^2$ where $(a+ib)^n=A+iB$.
If $n=2$, $x=a^2-b^2$, $y=2ab$, and $z=a^2+b^2$.
If $n=3$, $x=a^3-3ab^2$, $y=3a^2b-b^3$, and $z=a^2+b^2$, etc.

The solutions are easily written from the binomial expansion as the terms are alternately real and imaginary and the signs alternate by pairs of successive terms. When n is odd, the expansion has an even number of terms and x will have one term without any b and y will have one term without any a, and when n is even both of these terms are in x; hence, primitive solutions are derived when a and b are relatively prime and of different parity.

SOLUTIONS OF A TRIGONOMETRIC EQUATION

W. R. RANSOM, Tufts College

The solution given for

$$a\sin x + b\cos x = c$$

† This special case is discussed by this method in Agnew, R. P., Differential Equations, New York, McGraw Hill Book Company, 1942.

which appeared in Mattheson's note (*Rational Solutions of a Certain Trigonometric Equations*, this MONTHLY, vol. 55, p. 574 (1948)) avoids the extraneous answers that appear when either sine or cosine is expressed in terms of the other and the radical then removed by squaring. The introduction of an auxiliary angle is another means of avoiding extraneous roots; it is often used in numerical cases, and it may be of interest to observe this method of solution in general form.

Take h as the positive square root of a^2+b^2, and k as one square root of h^2-c^2. We can select a particular angle, M, for which $\sin M = b/h$ and $\cos M = a/h$. Upon dividing the given equation by h we obtain:

$$\cos M \sin x + \sin M \cos x = c/h$$

whence $\sin (M+x) = c/h$, and $\cos (M+x) = \pm k/h$. We can then determine $\sin x$ and $\cos x$ as follows:

$$\sin x = \sin [(M+x) - M] = \frac{c}{h}\cdot\frac{a}{h} - \frac{\pm k}{h}\frac{b}{h} = \frac{ac \mp bk}{a^2+b^2}$$

$$\cos x = \cos [(M+x) - M] = \frac{\pm k}{h}\frac{a}{h} + \frac{c}{h}\frac{b}{h} = \frac{bc \pm ak}{a^2+b^2}.$$

These values check in the equation, and there are no missing or extraneous roots. They also confirm the conclusion that the functions $\sin x$ and $\cos x$ are rational when k is rational, that is when $a^2+b^2=c^2+k^2$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 871. *Proposed by W. R. Ransom, Tufts College*

Is there any value of k other than $k=1$ for which $k \tan x \cot kx = 1$?

E 872. *Proposed by Leo Moser, University of Manitoba*

An enthusiastic problemist proposes at least one problem every day. In order not to overwork problem editors, however, he does not propose more than 730 problems a year. Given any positive integer n, show that he proposes exactly n problems in some set of consecutive whole days. (Dedicated to Victor Thébault.)

E 873. *Proposed by I. M. Hostetter, Oregon State College*

Let a vector be interpreted as a directed line segment and let $A * B$ be a symbolic product of two arbitrary vectors A and B such that (1) $A * B$ is a scalar, (2) $A * B = B * A$, (3) $A * (B+C) = A * B + A * C$. Is there an interpretation of $A * B$ other than the familiar inner, or scalar, product: $|A||B| \cos \theta$, where $|A|$, $|B|$ are the (positive) lengths of A and B, and θ is the angle between A and B?

E 874. *Proposed by C. S. Ogilvy, Trinity College*

If $y_1 = x^{1/2}$, $y_2 = (x+y_1)^{1/2}$, $\cdots$, $y_m = (x+y_{m-1})^{1/2}$, prove that $\lim_{m\to\infty} y_m$ is integral if and only if x is of the form $n(n-1)$, $n = 2, 3, \cdots$, in which case the limit is n.

E875. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A life-buoy is formed by two right circular cones joined by their common base. (1) Show that the centroid of the solid coincides with the centroid of the vertices of the quadrangle obtained by cutting the surface of the buoy by an axial plane. (2) Determine the ratio of the altitudes of the two cones in order that the buoy may be in equilibrium when it rests, on a horizontal plane, on one of the two surfaces, or indifferently on one or the other of the two surfaces, of the two cones.

SOLUTIONS

Perimeters of Pythagorean Triangles

E 828 [1948, 427]. *Proposed by Fritz Herzog, Michigan State College*

For positive integral x, let $P(x)$ denote the number of distinct primitive Pythagorean triangles of perimeter x. Show that $P(x)$ is unbounded; in fact, for any given non-negative integer r, the equation $P(x) = r$ has an infinitude of solutions. (Compare Problem E 812 [1948, 248].)

I. *Solution by the Proposer.* Let the sides of any primitive Pythagorean triangle be written in the form $2st$, s^2-t^2, s^2+t^2, where s, t are integers with $s>t>0$, $(s, t) = 1$, and $s \not\equiv t \pmod 2$. The correspondence between the primitive Pythagorean triangles and the pairs s, t is bi-unique. (See Hardy and Wright, *An Introduction to the Theory of Numbers*, Theorem 225, p. 189.) The perimeter of the triangle is then equal to $2su$, where $u = s+t$. The above conditions on s and t read, in terms of s and u,

$$(1) \qquad s < u < 2s, \qquad (s, u) = 1, \qquad u \equiv 1 \pmod 2.$$

Thus we obtain $P(x) = 0$ for odd x and assume from now on that x is even ($x = 2y$). According to the above, $P(2y)$ equals the number of those divisors u of y which satisfy (1), s being equal to y/u. If, in particular, y is odd and square-free then the second and third conditions of (1) will be satisfied for all divisors u of y so that $P(2y)$ equals the number of those divisors u of y for which, with $s = y/u$,

$$1 < u/s < 2. \tag{2}$$

We shall now exhibit an infinitude of odd squarefree y with $P(2y)=r$, for any given $r \geqq 0$. We denote by q_n the nth odd prime, and we make use of the relation

$$\lim_{n\to\infty} (q_{n+1}/q_n) = 1. \tag{3}$$

(See Hardy and Wright, *loc. cit.*, Theorem 8, p. 10.)

It is obvious that $P(2q_n)=0$ for $n \geqq 1$. Also $P(2q_nq_{n+1})=1$ if $q_{n+1}/q_n<2$, which by (3) is true for sufficiently large n. To solve $P(2y)=2$ we put $y=3\cdot 5\ q_nq_{n+1}$, where $q_{n+1}/q_n<6/5$, which by (3) holds for sufficiently large n. The inequalities $3\cdot 5q_n/q_{n+1}>3\cdot 5^2/6>2$, $q_nq_{n+1}/3\cdot 5>5\cdot 7/3\cdot 5>2$, and $1<5^2/3\cdot 6<5q_n/3q_{n+1}<5q_{n+1}/3q_n<5\cdot 6/3\cdot 5=2$ show that among the divisors u of y only $u=5q_n$ and $u=5q_{n+1}$ satisfy (2). Hence $P(2\cdot 3\cdot 5\ q_nq_{n+1})=2$. Finally, in order to solve $P(2y)=r$, where r is a given integer $\geqq 3$, we put $y=q_nq_{n+1}\cdots q_{n+r-1}Q$, where Q is a prime such that

$$\frac{q_{n+1}q_{n+2}\cdots q_{n+r-1}}{q_n} < Q < \frac{2q_nq_{n+1}\cdots q_{n+r-2}}{q_{n+r-1}}. \tag{4}$$

Such a prime Q exists for sufficiently large n because, as $n\to\infty$, the outer members of (4) approach ∞ (since $r\geqq 3$) and their ratio, *viz.* $2(q_n/q_{n+r-1})^2$, approaches 2 by repeated application of (3); from that the existence of Q for sufficiently large n follows by again applying (3). From (4) we conclude first that $Q>q_{n+r-1}$ (since $r\geqq 3$) so that y is odd and squarefree. Secondly, (4) implies the following inequalities:

$$\frac{q_nq_{n+1}\cdots q_{n+r-1}}{Q} > \frac{q_{n+r-1}^2}{2} > 2,$$

$$\frac{q_nq_{n+1}Q}{q_{n+2}q_{n+3}\cdots q_{n+r-1}} > q_{n+1}^2 > 2,$$

$$1 < \frac{q_nQ}{q_{n+1}q_{n+2}\cdots q_{n+r-1}} < \frac{q_{n+r-1}Q}{q_nq_{n+1}\cdots q_{n+r-2}} < 2.$$

These inequalities show that the only divisors u of y satisfying (2) are $u=q_{n+j}Q$, with $j=0, 1, \cdots, r-1$. This shows that $P(2q_nq_{n+1}\cdots q_{n+r-1}Q)=r$ and completes the proof.

II *Solution by Leo Moser, University of Manitoba.* $P(x)$ is equal to the number of different factorizations of $x/2$ into two factors m and n, $(m, n)=1$, $m<n<2m$. Such a factorization will be called "favorable."

Let p_i denote the ith prime. An elementary consequence of the prime number theorem is that for any $e>0$ there exists a j such that for $i>j$, $p_i \div p_{i+1}>1-e$.

Now take $r>2$ and consider $x=p_i^{r-2}p_{i+1}p_{i+2}\cdots p_{i+r}$. It is clear that for e sufficiently small the only "favorable" factorizations of x are

$$x/2 = (p_i^{r-2}p_{i+k})(p_{i+1}p_{i+2}\cdots p_{i+r}/p_{i+k}), \qquad k = 1, 2, \cdots, r.$$

Hence $P(x)=r$, and since i is restricted only by the condition $i>j$, there exist infinitely many solutions of $P(x)=r$, $r>2$.

To handle the case $r=2$ we again invoke the prime number theorem, which guarantees the existence of infinitely many sets of four primes which are "almost" proportional to 2, 3, 4, 5. With $x/2$ the product of four such primes $P(x)=2$.

Also solved by L. J. Burton.

Sufficient Condition for Continuity in an Interval

E 829 [1948, 427] *Proposed by S. H. Gould, Purdue University*

Let $f(x)$ be defined in a closed interval $[a, b]$ and have the property of assuming, in any subinterval $[c, d]$, every value between $f(c)$ and $f(d)$ Prove that $f(x)$ is continuous if no rational value is infinitely often assumed.

Solution by the Proposer. Suppose $f(x)$ to be discontinuous at x_0, say from above on the left. Then, for some $\epsilon>0$, we may choose $x_1<x_2<\cdots<x_n<\cdots\rightarrow x_0$ such that $f(x_n)>f(x_0)+\epsilon$ for all x_n. Let m be rational between $f(x_0)$ and $f(x_0)+\epsilon$. Take y_1 in (x_1, x_0) with $f(y_1)=m$, then $x_{i_1}>y_1$ and y_2 in (x_{i_1}, x_0) with $f(y_2)=m$, and similarly for $y_3, y_4, \cdots$. Then $f(x)$ assumes the rational value m infinitely often in (x_1, x_0), which is contrary to hypothesis.

Also solved by L. J. Burton, A. S. Day, Leo Moser and Alex Tytun.

Rider to I-47

E 831 [1948, 498]. *Proposed by K. W. Crain, Purdue, University*

If squares be constructed on the legs of a right triangle, the lines (which do not lie along the sides of the triangle) drawn from each end of the hypotenuse to a vertex of the opposite square intersect on the altitude which passes through the vertex of the right angle.

I. *Solution by Ragnar Dybvik, Levanger, Norway.* Let the right triangle, placed on a rectangular coordinate system, have its vertices at the points $A(0, b)$, $B(a, 0)$, $C(0, 0)$. The concerned vertices of the squares are the points $D(b, -b)$ and $E(-a, a)$. Let CH be the altitude on the hypotenuse. The equations of the lines BD, EA, CH are found to be

$$\begin{aligned}(a+b)x + ay - ab &= 0,\\ -bx - (a+b)y + ab &= 0,\\ -ax + by &= 0.\end{aligned}$$

Since the sum of the left members of these equations is identically zero it follows that the three lines are concurrent.

II. *Solution by W. B. Clarke, San Jose, California.* Using the notation of solution I and designating the intersections of CA and DB, CB and EA by M, N it is easily shown that $CM/MA = a/b$, $BN/NC = a/b$, $AH/HB = b^2/a^2$. Therefore $(BN)(CM)(AH) = (NC)(MA)(HB)$ and AN, BM, CH are concurrent by Ceva's theorem.

Also solved by Louis Berkofsky, Sheldon Best, W. E. Buker, L. J. Burton, W. J. Cherry, R. L. Clayton, A. C. Cohen, Jr., William Douglas, Gertrude Ehrlich, B. K. Gold, Frank Harary, Frank Herlihy, Banesh Hoffmann, R. E. Horton, S. J. Jasper, L. M. Kelly, Roger Lessard, Marie Madden, Donald Marsh, Eugene McLachlan, R. D. McWilliams, Leo Moser, C. S. Ogilvy, W. O. Pennell, O. M. Rasmussen, C. C. Richtmeyer, P. D. Thomas, W. I. Thompson, C. W. Trigg (three ways), E. H. Vance, Margaret Willerding, Maud Willey, J. E. Winter, the proposer, and R. V. Andree's freshman engineering class at University of Wisconsin.

Editorial Note. Trigg pointed out that this problem has occurred three times in *School Science and Mathematics*: as problem 563, Mar. 1919; problem 841, Dec. 1924; problem 1651, May 1940. In addition to proofs similar to those given above there appear some using only high school geometry. One proof shows that the three lines in question are the altitudes of triangle ABT, where T is the intersection of the remote sides of the squares prolonged.

No one observed that the theorem is true for *any* triangle. In fact we have: *Given any triangle ABC with squares $ADFC$, $BEGC$ described either both externally or both internally on the sides CA and CB, then AE, BD intersect on the altitude through C and AG, BF intersect on DE.* For the first part of the theorem we may replace the squares by two similar rectangles.

Rational Points on a Circle

E 832 [1948, 498]. *Proposed by V. E. Dietrich, Purdue University*

If a circle has a center with at least one irrational coordinate, then there are at most two points on the circle with rational coordinates.

Solution by C. S. Ogilvy, Trinity College. Suppose there are three points on the circle, each with rational coordinates. Then the equation of the circle can be found by substituting the coordinates of the three points, one pair after another into the equation

$$x^2 + y^2 + Ax + By + C = 0.$$

The resulting three simultaneous linear equations have rational coefficients, and hence yield rational values for A, B, C. This contradicts the hypothesis, since $-A/2$ and $-B/2$ are the coordinates of the center of the circle.

Also solved by F. Bagemihl, Murray Barbour, L. J. Burton, B. B. Dressler, N. J. Fine, L. B. Hedge, L. M. Kelly, Roger Lessard, Norman Miller, Leo Moser, S. T. Packer, C. F. Pinzka, C. W. Trigg, Alex Tytun, and the proposer.

Miller gave the following examples. The circle $(x-\sqrt{2})^2+y^2=3$ has on it the rational points (0, 1) and (0, −1); $(x-\sqrt{2})^2+y^2=3-2\sqrt{2}$ has the single rational point (1,0); $(x-\sqrt{2})^2+y^2=\sqrt{3}$ has no rational point. The circle $(x-\sqrt{2})^2+(y-\sqrt{2})^2=6$ has on it the rational points (1, −1) and (−1, 1); $(x-\sqrt{3})^2+(y-\sqrt{2})^2=5$ has the single rational point (0, 0); $(x-\sqrt{3})^2+(y-\sqrt{2})^2=4$ has no rational point.

R. W. Hamming pointed out that the words "irrational" and "rational" can be replaced by "transcendental" and "algebraic" respectively.

A City Park

E 833 [1948, 498]. *Proposed by P. D. Thomas, Washington, D. C.*

A surveyor in laying out a square park area in a city found it was necessary because of obstructions to shorten two opposite sides by one foot, but both the length and width were in integral feet. To check his work he ran a diagonal of the resulting rectangle. Imagine his surprise to find that the semi-perimeter (diagonal+length+width) was in integral rods! What were the dimensions of the field?

Solution by C. W. Trigg, Los Angeles City College. If the sides and diagonal of the park are x, $x-1$, y feet, respectively, then

$$2(y + 2x - 1) = 33s,$$

where s is an integer. Also $y^2=x^2+(x-1)^2$, which may be written as

$$2y^2 = (2x - 1)^2 + 1.$$

Hence we seek a solution of the Pellian equation $w^2-2y^2=-1$ such that $w+y$ is a multiple of 33. Two consecutive solutions of this equation are $(w_0, y_0) = (-1, 1)$ and $(w_1, y_1) = (1, 1)$. All other solutions are given by the recurrence formulas

$$w_{n+2} = 6w_{n+1} - w_n, \qquad y_{n+2} = 6y_{n+1} - y_n.$$

The two smallest solutions such that $w+y\equiv 0 \pmod{33}$ are

$$(8119, 5741) \text{ and } (318281039, 225058681).$$

Since the latter solution corresponds to a park of side 159140520 feet, it represents an impossible situation. Therefore the smaller pair provides the unique solution in which the sides and diagonal of the park are 4060, 4049, 5741 feet, respectively, the semi-perimeter being 840 rods.

Also solved by Murray Barbour, E. M. Berry, L. J. Burton, John Cromelin, Monte Dernham, William Douglas, G. B. Huff, Roger Lessard, Leo Moser, Leola Odland, C. S. Ogilvy, S. T. Parker, Alex Rosenberg, C. M. Sandwick, W. R. Talbot, John Walker, and the proposer.

Moser said that it can be shown, from the recurrence formulas, that the semi-perimeter will be divisible by 33 if and only if n is divisible by 6.

The problem of finding Pythagorean triangles whose legs are consecutive integers is considered by Kraitchik, *Mathematical Recreations*, p. 100.

Fibonacci Numbers as Determinants

E 834 [1948, 498]. *Proposed by Don Walter, Pomona College*

Show that

$$F_n = \begin{vmatrix} 1 & -1 & 1 & -1 & 1 & -1 \cdots \\ 1 & 1 & 0 & 1 & 0 & 1 \cdots \\ 0 & 1 & 1 & 0 & 1 & 0 \cdots \\ 0 & 0 & 1 & 1 & 0 & 1 \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \cdots \end{vmatrix},$$

where F_n is the nth term of the Fibonacci sequence 1, 1, 2, 3, 5, $\cdots$, x, y, $x+y$, $\cdots$ and the determinant is of order $n-1$.

Solution by Alex Tytun, New York, N. Y. Denoting the above determinant by D_n, it is seen that $D_2=1$, $D_3=2$. It remains to show that $D_n=D_{n-1}+D_{n-2}$, $n\geqq 4$. In D_n subtract the $(n-3)$th column from the $(n-1)$th, the $(n-4)$th from the $(n-2)$th, $\cdots$, the first from the third, obtaining

$$D_n = \begin{vmatrix} 1 & -1 & 0 & 0 & 0 & 0 \cdots \\ 1 & 1 & -1 & 0 & 0 & 0 \cdots \\ 0 & 1 & 1 & -1 & 0 & 0 \cdots \\ 0 & 0 & 1 & 1 & -1 & 0 \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \cdots \end{vmatrix}.$$

By expanding this determinant with reference to the first row, there results the desired relation.

Also solved by L. J. Burton, R. G. Buschman, Gertrude Ehrlich, Roger Lessard, Donald Marsh, Leo Moser, S. T. Parker, C. F. Pinzka, C. W. Trigg, and the proposer.

Recurrence Relations Defining Null Sequence

E 835 [1948, 498]. *Proposed by Kenneth May, Carleton College*

If x_1, y_1, and a are real numbers, and for all integral $n\geqq 1$ we have $x_{n+1}=a(x_n^2-y_n^2)$ and $y_{n+1}=2ax_ny_n$, for what values of x_1 and y_1 will $\lim_{n\to\infty} x_n=\lim_{n\to\infty} y_n=0$?

Solution by Norman Miller, Queen's University, Ontario. Represent the point (x_1, y_1) by the polar coordinates (r, θ) and suppose $a\neq 0$. By applying the given formulas we find that the point (x_{n+1}, y_{n+1}) is, in polar coordinates, $(a^{2^n-1}r^{2^n}, 2^n\theta)$. Hence, if $(x_1^2+y_1^2)^{1/2}=r<|a|^{-1}$, the points (x_n, y_n) approach the origin as a limit as n increases; if $r=|a|^{-1}$, the points remain on the circle

$r=|a|^{-1}$ and, in general, approach no limit; if $r>|a|^{-1}$, the points recede indefinitely from the origin.

Also solved by L. J. Burton, B. B. Dressler, N. J. Fine, W. Fulks, J. F. Heyda, Roger Lessard, Donald Marsh, S. T. Parker, C. F. Pinzka, and the proposer.

Chains of Circles

E 840 [1948, 576]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Inscribe three equal circles (A), (B), (C) in the corresponding interior angles of a triangle ABC such that we may insert between (B) and (C) a chain of tangent circles equal to (B) and (C) and all touching side BC, and similar chains between (C) and (A) and between (A) and (B). What is the condition for possibility of solution, and how many solutions are there for a given triangle ABC?

Solution by C. M. Sandwick, High School, Easton Pa. Suppose the problem solved and that A', B', C' are the centers of circles (A), (B), (C). Clearly, the sides of triangle $A'B'C'$ are commensurable, and therefore, since triangles $A'B'C'$ and ABC are similar, the sides of triangle ABC must be commensurable.

On the other hand, suppose the sides a, b, c of triangle ABC are commensurable, m being a common unit of measure. Then there are positive integers, p, q, r such that $a=pm$, $b=qm$, $c=rm$. Let I be the incenter of ABC and draw $BD=m/2$ perpendicular to BC on the opposite side of BC from A. Let DI cut BC in E, and let the perpendicular to BC at E cut BI in B'. Draw triangle $A'B'C'$ with sides parallel to those of ABC, A', B', C' falling on AI, BI, CI, respectively. Let the sides of $A'B'C'$ be denoted by a', b', c' and let $B'E$ be denoted by t. Then, from similar triangles, it is easy to show that $a'/2t=a/m=p$. Similarly, $b'/2t=q$ and $c'/2t=r$. Therefore there exists a solution to the problem such that there are $p+1$ circles in the chain along side BC and having centers on $B'C'$, $q+1$ circles in the chain along CA and having centers on $C'A'$ and $r+1$ circles in the chain along AB and having centers along $A'B'$.

If one solution exists, then, of course, there are infinitely many solutions, obtained by taking $m=g/k$, where g is the greatest common measure of a, b, c and k is any positive integer.

Also solved by P. R. Beesack, Richard Courter, C. F. Pinzka, and the proposer.

Folding an Envelope into Tetrahedra

E 841 [1948, 640]. *Proposed by C. W. Trigg, Los Angeles City College*

(a) Can any sealed rectangular envelope, after a single straight cut, be folded into two congruent tetrahedra? Will the position of the cut affect the size of the tetrahedra?

(b) How should the cut be made to make the total number of folds and unfolds a minimum?

(c) What should be the relative dimensions of the envelope in order that the tetrahedra be regular?

Solution by the Proposer. (a) Since the areas of congruent tetrahedra are equal, the envelope must be cut so that the two pieces will have equal areas. Hence the cut must pass through the center of the rectangle.

Let A, B, C, D be the corners of the envelope, with AD the shorter side, K the center of the front, K' the center of the back, EKF parallel to AD and GKH parallel to BA. The corresponding lines on the back together with the diagonals divide the surface of the envelope into sixteen congruent triangles, such that the adjacent angles at the corners and the center are alternately equal and complementary, the adjacent angles at the center of the edges are right angles, and the adjacent triangles are symmetric with reference to those common sides which are edges or are parallel to edges. (To facilitate future folding, fold the envelope along the diagonals and along EF before cutting.)

The cut MKN may be made in four essentially different ways:

1. Cut along line EKF. Unfold AE, AD, DF. Fold out along AK, AK', DK, DK'. Then KEK', and KFK' fold out into straight lines forming triangles AKK' and DKK' which are congruent to ADK and ADK'. Thus an isosceles tetrahedron is formed in three unfolds and four folds. The other half of the envelope can be folded similarly to give a tetrahedron congruent to the first one.

2. Cut through K at an angle to EF so that M falls on AB and N on DC. Then triangles MEK, MEK', NFK, NFK' are congruent. Hence, when the half-envelope is unfolded and folded as in way 1, an additional fold along KEK' will bring M into coincidence with N completing the fourth face DKK' of a tetrahedron congruent to those in way 1. (Three unfolds, five folds.)

3. Cut along BDK. Unfold AD and AB. Fold out AK and AK' and fold in along KEK'. Thus with two unfolds and three folds a tetrahedron congruent to those in way 1 is secured.

4. Cut through K so that M falls on BC and N on AD. Then triangles MGK, MGK', NHK, NHK' are congruent. Unfold AN, AB, BM. Fold out along AK, AK', BK, BK' until KEK' is a straight line. When the figure is folded in along KK', triangles ANK and BMK join and triangles ANK' and BMK' join to form faces congruent to AKK' and BKK' and to ADK. Hence this tetrahedron is congruent to those in way 1. (Three unfolds, five folds.)

Therefore the size of the tetrahedra is independent of the direction of the cut.

A necessary condition that these tetrahedra be formed is that the angles formed at a corner of the envelope by a diagonal with the edges be unequal. Thus the *single exception* is that of a square envelope for which angles EAK, EAK', HAK, HAK' are equal to 45°. When the cutting, unfolding, and folding of a square envelope is done in any one of the four ways, instead of a tetrahedron, two superposed squares are secured. That is, a square envelope after a single cut may be folded into two square envelopes.

(b) The cut should be made along a diagonal (way 3) to minimize the unfolding and folding.

(c) If the tetrahedron is to be regular, each face will be an equilateral tri-

angle, whence $AD/KH=2/\sqrt{3}$. Thus the required relative dimensions of the envelope are given by $AD/GH=AD/AB=1/\sqrt{3}$.

It may be observed further that if the envelope be slit along the edge, and then folded along the lines joining the midpoints of the slit edge to the other corners, a single isosceles tetrahedron is obtained in three unfolds and four folds.

Also solved by D. W. Matlack.

Cubics with Rational Zeros and Bend Points

E 842 [1948, 640]. *Proposed by H. L. Lee, University of Tennessee*

Determine cubic functions $f(x)$ for which $f(x)=0$ and $f'(x)=0$ have rational roots.

Solution by Alan Wayne, Flushing, N. Y. Let $f(x)=(x-a)(x-b)(x-c)$, where a, b, c are rational. If any two of the zeros, say a and b, are equal, then $f(x)$ and $f'(x)$ both have rational zeros for all rational a and c. Suppose, then, no two of a, b, c are equal. The substitution $y=(x-a)/(b-a)$ transforms $f(x)=0$ and $f'(x)=0$ into $F(y)=y(y-1)(y-q)=0$ and $F'(y)=3y^2-2(q+1)y+q=0$, where $q=(c-a)/(b-a)$. Now $f'(x)$ will have rational zeros if and only if $F'(y)$ has rational zeros, that is, if and only if

$$q^2 - q + 1 = r^2, \tag{1}$$

where r is rational. But r is rational if and only if there exists a rational number $v\neq -1$ such that $r=1/(v+1)$. Then there is a rational number $t\neq 0$ such that $q=(v/t+1)/(v+1)$. We also have $v\neq 0$ and $t\neq 1$, for otherwise $q=1$ and $b=c$. From (1), since $v\neq 0$, we find $v=-(1+t)/(t^2-t+1)$, and it further follows that $t\neq -1$, 2. If, in addition, $t\neq 1/2$, then $q=(t^2-2t)/(1-2t)$. Thus finally, since $c=(b-a)q+a$, we find that a necessary and sufficient condition for both $f(x)$ and $f'(x)$ to have *distinct* rational zeros is that

$$c = a(1 - t^2)/(1 - 2t) + b(t^2 - 2t)/(1 - 2t), \tag{2}$$

where a and b are any two distinct rational numbers and t is any rational number different from -1, 0, 1/2, 1, 2. Thus any two distinct rational numbers may be chosen for a and b and a corresponding c determined by (2). The cubic polynomial is then easily constructed.

Also solved by Joshua Barlaz, L. J. Burton, Roger Lessard, C. M. Sandwick, C. W. Trigg, and the proposer.

A related problem is E 328 [1939, 170].

Spheres Related to an Oblate Ellipsoid

E 843 [1948, 640]. *Proposed by P. D. Thomas, Washington, D. C.*

Show that the difference in the radii of the two spheres, one equivalent in area, the other equivalent in volume, to an oblate ellipsoid of revolution is of the fourth order in the eccentricity of the generating ellipse.

Solution by the Proposer. Let a, b, e where $b^2=a^2(1-e^2)$, be the usual parameters of the generating ellipse, and let r_a and r_v be the radii of the equivalent-area and equivalent-volume spheres, respectively. These two radii are then defined by

$$2r_a^2 = a^2 + b^2(\tanh^{-1} e)/e = a^2[1 + (1 - e^2)(\tanh^{-1} e)/e], \tag{1}$$

$$r_v^3 = a^2 b = a^3(1 - e^2)^{1/2}. \tag{2}$$

Now

$$\begin{aligned}\tanh^{-1} e &= (1/2) \ln [(1 + e)/(1 - e)] \\ &= e + e^3/3 + e^5/5 + e^7/7 + \cdots,\end{aligned}$$

whence, from (1),

$$\begin{aligned}r_a &= a[1 - e^2(1/3 + e^2/15 + e^4/35 + \cdots)]^{1/2} \\ &= a(1 - e^2/6 - 17e^4/360 - 67e^6/3024 - \cdots).\end{aligned}$$

From (2) we have

$$r_v = a(1 - e^2)^{1/6} = a(1 - e^2/6 - 5e^4/72 - 55e^6/1296 - \cdots)$$

It is thus seen that $r_a - r_v = a(e^4/45 + \cdots)$.

Also solved by L. J. Burton, Roger Lessard, Hyman Orlin, and C. W. Trigg.

Trigg showed that if we denote by R_a and R_v the radii of the spheres equivalent in area and volume, respectively, to the *prolate* ellipsoid, then $R_a - R_v$ is also of the fourth order in e.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4350. *Proposed by V. L. Klee, Jr., University of Virginia*

Money deposited in Postal Savings draws interest at the rate of $\frac{1}{2}\%$ quarterly, beginning with the month following that in which it is deposited. Interest is not compounded, and is not paid until the deposit is withdrawn. Under the simplifying assumption that deposits and withdrawals are permissible in any

amount whatever, find the optimum pattern of withdrawal and re-deposit for an initial deposit A, principal and interest to be permanently withdrawn at the end of n months.

4351. *Proposed by Albert Wilansky, Lehigh University, Bethlehem, Pa.*

Let $f(x, y)$ be continuous for all (x, y). On each circle with center at the origin f assumes a minimum at certain points. Is the set of all such points throughout the plane connected?

4352. *Proposed by Paul Erdös, Syracuse University*

Denote by $f(n; a_1, a_2, \cdots, a_k)$ the number of positive integers $m \leqq n$ which are either divisors or multiples of one of the a's $(a_i > 1)$. Prove that

$$f(n; a_1, a_2, \cdots, a_k) \leqq f(n; 2, 3, \cdots, p_k),$$

$2, 3, \cdots, p_k$ being the first k primes.

4353. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left[\frac{\log n}{\log 2}\right] = \gamma,$$

where $[x]$ denotes the integral part of x, and γ is Euler's constant.

4354. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A necessary and sufficient condition for a tetrahedron to be isosceles is that each of two bialtitudes of the tetrahedron divide the opposite edges proportionally.

SOLUTIONS

A Functional Equation

3649 [1933, 610]. *Proposed by F. T. O'Doubler, Springfield, Mo.*

Solve the functional equation

$$f(xy) = [f(x)]^{y^\beta}[f(y)]^{x^\beta},$$

where β is a real constant and $f(x)$ is a real continuous and single-valued function of the real variable x.

Solution by N. A. Court, University of Oklahoma. With $g(x) \equiv \log f(x)$, we have

$$g(xy) = y^\beta g(x) + x^\beta g(y). \tag{1}$$

For $y = x$, (1) becomes

$$g(x^2) = 2x^\beta g(x),$$

and by induction it is readily shown that the equality

$$(N) \qquad g(x^n) = nx^{(n-1)\beta}g(x)$$

is valid for all positive integral values of n.

For $y=1$ and $y=x^{-1}$, (1) yields respectively

$$g(1) = 0, \qquad g(x^{-1}) = -x^{-2\beta}g(x).$$

Thus formula (N) is valid for $n=0, -1$, and an easy induction completes the proof that it is valid for all integral values of n.

If in (1) we replace x and y by $x^{1/u}$ and $x^{(u-1)/u}$, respectively, where u is an integer, positive or negative, we have

$$(2) \qquad g(x) = x^{(u-1)\beta/u}g(x^{1/u}) + x^{\beta/u}g(x^{(u-1)/u}),$$

but, by virtue of (N),

$$g(x^{(u-1)/u}) = g\left[(x^{1/u})^{(u-1)}\right] = (u-1)x^{(u-2)\beta/u}g(x^{1/u}),$$

with which (2) is easily reduced to

$$(3) \qquad g(x^{1/u}) = u^{-1}x^{(1-u)\beta/u}g(x),$$

so that formula (N) is valid when n is a rational fraction with unit numerator.

Let v/u be a rational fraction. Since, from (N),

$$g(x^{v/u}) = g\left[(x^{1/u})^v\right] = vx^{(v-1)\beta/u}g(x^{1/u}),$$

we have, making use of (3),

$$g(x^{v/u}) = vx^{(v-1)\beta/u}\cdot(1/u)\cdot x^{(1-u)\beta/u}g(x) = (v/u)x^{(v-u)\beta/u}g(x).$$

Thus (N) is valid for all rational values of n.

An irrational number may be considered as the limit of an infinite sequence of rational numbers. Since the formula (N) is valid for all these rational numbers, it will remain valid for the limit, since the function f, and therefore also g, is continuous by hypothesis. Thus (N) is valid for all real values of n.

By differentiating both sides of (N) with respect to the parameter n we have

$$\frac{dg(x^n)}{d(x^n)}\cdot x^n \log x = x^{-\beta}g(x)(x^{\beta n} + n\beta x^{\beta n}\log x).$$

This equation is valid for all values of n, and in particular for $n=1$. Hence

$$\frac{dg(x)}{g(x)} = \frac{dx}{x\log x} + \frac{\beta dx}{x}.$$

By integration we obtain

$$\log g(x) = \log\log x + \beta\log x + \log c,$$

where c is an arbitrary constant. From this equation upon replacing $g(x)$ by $\log f(x)$, we derive the desired result,

$$f(x) = x^{(cx^\beta)}.$$

Tetrahedron, Hyperboloid, and Paraboloid

4197 [1946, 161]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron $T \equiv ABCD$ the perpendiculars from an arbitrary point P to the planes of the faces meet again the pedal sphere of P in A', B', C', D'. (1) The intersections of corresponding faces of T and $A'B'C'D'$ belong to a hyperboloid. Generalize. (2) If P is on a sphere with center O and if the perpendiculars to the faces meet it in A'', B'', C'', D'', the orthologic center of T and $A''B''C''D''$, other than P, is the focus of the inscribed paraboloid with the axis parallel to OP.

Solution by the Proposer. (1) If A_1, B_1, C_1, D_1 are the projections of P on the planes of the faces BCD, CDA, DAB, ABC of T, we have, taking into account both magnitudes and directions,

$$(PA_1)(PA') = (PB_1)(PB') = (PC_1)(PC') = (PD_1)(PD') = k^2, \quad \text{say,}$$

whence T and $A'B'C'D'$ are polar reciprocals with respect to the sphere with center P and radius k (k real or imaginary). Therefore the intersections of corresponding faces of T and $A'B'C'D'$ belong to a hyperboloid.

(2) The planes p_a, p_b, p_c, p_d drawn through A'', B'', C'', D'' perpendicular to PA'', PB'', PC'', PD'' are concurrent at the point P' on the sphere (O) diametrically opposite to P.

By a translation equal and parallel to $P'A$, the planes p_b, p_c, p_d may be made to coincide with the planes CDA, DAB, ABC. If P goes into P_1 and if B_1'', C_1'', D_1'' are the projections of P_1 on CDA, DAB, ABC, then B_1'', C_1'', D_1'' correspond to B'', C'', D'' under the translation. Therefore the perpendicular through A to the plane $B''C''D''$ is also perpendicular to the plane $B_1''C_1''D_1''$, and we see that this perpendicular is the isogonal of line AP_1 for the trihedral $A-BCD$.

The same reasoning applied to vertices B, C, D shows that the orthologic center Q obtained by drawing perpendiculars through A, B, C, D to the planes $B''C''D''$, $C''D''A''$, $D''A''B''$, $A''B''C''$ is the isogonal conjugate, with respect to T, of the point at infinity in the direction OP. The projections of Q on the faces of T are therefore coplanar and Q is thus the focus of the paraboloid of revolution inscribed in T and having its axis parallel to OP.

Also solved by M. R. Bouvaist, who used analytical methods for part (2).

Editorial Note. For authorities for certain parts of the above proof see exercise 35(b), p. 213, and articles 741 and 748 in N. A. Court's *Modern Pure Solid Geometry*.

The analogous theorems for the plane are also true, a line of collinearity corresponding to the hyperbolic group of lines in part (1).

Orthocentric Tetrahedron

4219 [1946, 471]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In an orthocentric tetrahedron $ABCD$ with the altitudes AA', BB', CC', DD', let H' be the inverse of the orthocenter H with respect to the circum-

sphere, which the lines $H'A$, $H'B$, $H'C$, $H'D$ meet again in A_1, B_1, C_1, D_1. Show that the tetrahedrons $A_1B_1C_1D_1$ and $A'B'C'D'$ are similar and that the volume of the first is 27 times that of the second.

*Solution by the Proposer.** This problem is a particular case of the following general proposition (which may be new):

With the vertices of a tetrahedron $T \equiv ABCD$ as centers we construct arbitrary spheres (A, m), (B, n), (C, p), (D, q). The six spheres of similitude of these four spheres taken in pairs have in common two points V and W such that $OV \cdot OW = R^2$, R being the radius of the circumsphere (O) of T.† The lines AV, BV, CV, DV intersect (O) again in A_1, B_1, C_1, D_1 and the lines AW, BW, CW, DW meet (O) again in A_2, B_2, C_2, D_2. The tetrahedrons $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are equal, since V and W are inverse with respect to (O).

In the present problem six spheres of similitude passing through H and H' may be constructed since the Newtonian plane of the spheres $(A, m), \cdots,$ is the perpendicular bisector of the segment HH'. By Neuberg's Theorem the squares of the radii of $(A, m), \cdots,$ are determined to within a factor of proportionality.‡ The corresponding tetrahedrons $A_2B_2C_2D_2$ and $A_1B_1C_1D_1$ are equal. The original result then follows since $A_2B_2C_2D_2$ is the homothetic transform $(H, 3)$ of the tetrahedron $A'B'C'D'$.§

A Divisibility Problem

4267 [1947, 479]. *Proposed by C. F. Pinzka, Student, Rutgers University*

Let p be a prime greater than 3, and let r/ps be the sum of the harmonic series, $1+1/2+1/3+\cdots$ to p terms. Prove that p^3 divides $r-s$.

I. *Solution by Ernst Trost, Technikum Winterthur, Zürich, Switzerland.* From Fermat's theorem follows

$$(1) \qquad \begin{aligned} x^{p-1} - 1 &\equiv (x-1)(x-2)\cdots(x-p+1) \\ &= x^{p-1} - s_1x^{p-2} + \cdots - s_{p-2}x + s_{p-1} \pmod{p}, \end{aligned}$$

and hence we have

$$\begin{aligned} s_{p-1} &= (p-1)! \equiv -1 \pmod{p} && \text{(Wilson's theorem)} \\ s_{p-2} &\equiv [1 + 1/2 + \cdots + 1/(p-1)](p-1)! \equiv 0 \pmod{p} \\ s_{p-3} &\equiv 0 \pmod{p}. \end{aligned}$$

From (1) with $p>3$, we have

$$(p-1)! \equiv s_{p-3}p^2 - s_{p-2}p + s_{p-1} \pmod{p^3},$$

and thus

$$s_{p-2} \equiv ps_{p-3} \equiv 0 \pmod{p^2}.$$

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* Translated and references supplied by W. E. Byrne, Virginia Military Institute.

† Court, Modern Pure Solid Geometry, p. 204.

‡ Court, loc. cit., p. 203.

§ Court, loc. cit., p. 265.

Now, if

$$1+\frac{1}{2}+\cdots+\frac{1}{p-1}+\frac{1}{p}=\frac{r}{ps},$$

we obtain

$$s_{p-2}=\frac{(p-1)!}{s}\cdot\frac{r-s}{p}\equiv 0 \pmod{p^2}.$$

The integer $(p-1)!/s$ is prime to p, whence

$$r-s\equiv 0 \pmod{p^3}.$$

II. *Solution by Colin Blyth, Jr., Student, University of North Carolina.* We have at once

$$\sum_{a=1}^{p-1} a^{-1}=(r-s)/ps,\ (s,\ p)=1.$$

But $\sum_{a=1}^{p-1} a^{-1}\equiv 0 \pmod{p^2}$ for all prime >3 is a known theorem. (Wolstenholme's. L. E. Dickson, *History of the Theory of Numbers*, v. 1, p. 96, gives fifteen references, including this MONTHLY, problem 216 [1915, 103].) Finally $(r-s)/ps\equiv 0 \pmod{p^2}$ implies $r-s\equiv 0 \pmod{p^3}$, which is the desired result.

Also solved by P. T. Bateman, Alfred Brauer, D. H. Browne, P. A. Clement, R. E. Crane, Free Jamison, Y. S. Luan, Leo Moser, C. D. Olds, Chih-yi Wang, and the Proposer.

Special Properties of 1947

4269 [1947, 480]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find all numbers $N=abcd$ of four distinct digits, zero excluded, such that the sum

$$ab+ac+ad+bc+bd+cd$$

of products of the digits two at a time shall equal the sum

$$a^2+b^2+c^2+d^2$$

of the squares of the digits. For which among these is it true that the sum of the two-digit numbers ab and cd equals

$$a^2+c^2+d^2?$$

Solution by C. W. Trigg, Los Angeles City College.

1) If four digits are to satisfy the relation $\Sigma a^2=\Sigma ab$, then all must be odd, all must be even, or three must be odd and one even.

(2) $$(\sum a)^2=\sum a^2+2\sum ab=3\sum a^2.$$

Therefore $\Sigma a \equiv 0 \pmod 3$, and

$$\sum a^2 \equiv 0 \pmod 3. \tag{3}$$

4) Thus one digit is a multiple of 3 and the others are congruent to each other (mod 3).

There are only two sets of four distinct digits satisfying conditions (1) and (4). These two, 1347 (for which $\Sigma a^2 = \Sigma ab = 75$) and 1479 (for which $\Sigma a^2 = \Sigma ab = 147$), may be permuted to give twenty-four numbers each. These permutations constitute the only values of N.

If $10a+b+10c+d = a^2+c^2+d^2 = M$, then $9(a+c)+\Sigma a+b^2 = \Sigma a^2$. Therefore b is a multiple of 3. For the 1347 set, $b=3$, $\Sigma a^2 = 75$ and $M = 66$, so that $d=3$, which is impossible. For the 1479 set, $b=9$, $M=66$, so that $d=7$. Hence the two integers which meet both conditions of the problem are 4917 and 1947, which latter is the year of the proposal.

Also solved by Colin Blyth, Jr., J. S. Frame, N. G. Gunderson, Free Jamison, Roger Lessard, Ou Li, Leo Moser, M. A. Rousseau, Chih-Yi Wang, and the Proposer.

Editorial Note. The above solution is adequate for the proposed problem about digits. Considerable interest, however, has developed in regard to a general solution of the equation

$$a^2 + b^2 + c^2 + d^2 = ab + ac + ad + bc + bd + cd. \tag{i}$$

(See Problem E. 823, [1949, 185].) Solving (i) as a quadratic in d, we have

$$2d = a + b + c + 3u$$

where

$$9u^2 = (a + b + c)^2 - 4(a^2 + b^2 + c^2 - ab - ac - bc),$$

or

$$4ab = 3u^2 + (a + b - c)^2.$$

This last equation can be put in the form

$$c = a + b + v, \qquad v^2 + 3u^2 = 4ab,$$

whence u, v are of the same parity, and may have any sign. Putting $m = (u+v)/2$, $n = (u-v)/2$, we have the result: if a, b, c, d are integers satisfying (i), then there exist integers m, n such that

$$ab = m^2 + mn + n^2, \tag{ii}$$

$$c = a + b + m - n, \qquad d = a + b + 2m + n. \tag{iii}$$

Conversely, m and n being arbitrary integers, if a and b are chosen to satisfy (ii) then they, together with c and d, given by (iii), satisfy (i).

From the known properties of numbers of the form p^2+pq+q^2 (therefore of form x^2+3y^2) we note from (ii) that a and b, apart from their highest common factor, must be also of the same form. Since a, b are any two of a, b, c, d, it follows that all four of them, except for any factors common to all, are of the form p^2+pq+q^2, where p and q are integers, positive, negative or zero.

Multiples of Irrational Numbers

4270 [1947, 549]. *Proposed by S. H. Gould, Victoria College, Toronto*

Let b be a fixed integer, $m=1, 2, \cdots$, and q irrational with $0<q<1$. Call the interval $(m, m+1)$ a gap if it does not contain a multiple of $b+q$. Prove that every set of b successive gaps contains exactly one multiple of $1+b/q$.

Solution by Guy Stevenson, University of Louisville, Kentucky. For every integer r there exists an integer x such that one or the other of the following relations holds:

$$rq < x < (r+1)q, \tag{1}$$

$$x < rq < (r+1)q < x+1. \tag{2}$$

If (1) holds, we have

$$r(b+q) < rb+x, \qquad rb+x+b < (r+1)(b+q)$$

so that the interval $I \equiv [r(b+q), (r+1)(b+q)]$ contains the b gaps corresponding to $m=rb+x, rb+x+1, \cdots, rb+x+b-1$.

On the other hand, if (2) holds, we have

$$rb+x < r(b+q), \qquad (r+1)(b+q) < rb+x+b+1$$

so that the interval I contains the $b-1$ gaps corresponding to $m=rb+x+1, rb+x+2, \cdots, rb+x+b-1$.

Thus (1) is the necessary and sufficient conditions that the interval I correspond to a set of b successive gaps. Now (1) implies also that

$$r(b+q) < \frac{x(b+q)}{q} < (r+1)(b+q)$$

which is the desired conclusion. It is seen also that no multiple of $1+b/q$ exists in an interval I which contains only $b-1$ gaps.

Also solved by Free Jamison, Leo Moser, and the Proposer.

Tetrahedron and Hyperbolic Set of Lines

4271 [1947, 549]. *Proposed by N. A. Court, University of Oklahoma*

The external bisectors of the three face angles of each trihedron of a given tetrahedron are coplanar. The four planes form a second tetrahedron. Show that the lines joining corresponding vertices of the two tetrahedrons form, in general, a hyperbolic group.

Solution by Ou Li, Yenching University, Peiping, China. Let $ABCD$ and

$A'B'C'D'$ be the given first and second tetrahedrons. Construct the ex-centers I_1, I_2, I_3 of the triangle ABC. Since I_2I_3 is one of the external bisectors of three face angles of the trihedron at A, it lies on the face $B'C'D'$ of the second tetrahedron $A'B'C'D'$. Similarly I_3I_1 and I_1I_2 lie respectively on the faces $C'A'D'$ and $A'B'D'$. It follows that I_1 lies on $A'D'$, I_2 on $B'D'$ and I_3 on $C'D'$. We note that AI_1, BI_2 and CI_3 are concurrent at the incenter I of the triangle ABC. Thus the sets of points A, A', I, I_1, D'; B, B', I, I_2, D'; C, C', I, I_3, D' are respectively coplanar, so that the line $D'I$ intersects AA', BB', and CC', and evidently also DD'.

Analogously, also $C'I$, $B'I$ and $A'I$ intersect AA', BB', CC' and DD'. Thus, the four lines AA', BB', CC', DD' which are met by $A'I$, $B'I$, $C'I$, $D'I$ form, in general, a hyperbolic group of lines.

Also solved by the Proposer who points out that if the tetrahedron $ABCD$ is isodynamic, then AA', BB', CC', DD' are concurrent.

A Summation Trick

4272 [1947, 549]. *Proposed by A. L. Epstein, Asbury Park, N. J.*

Given the sequence n_i where $n_1=a$, $n_2=b$, and $n_{i+2}=n_{i+1}+n_i$, $i=1, 2, \cdots$. Show that

$$\sum_{i=1}^{4k-2} n_i = r_k n_{2k+1}$$

and determine the form of r_k. (This is suggested by the familiar trick in which the subject selects n_1, n_2, and computes n_3, n_4, $\cdots$, n_7. He tells the operator the result n_7, and continues the calculation through n_{10}. When he adds the ten numbers he finds the operator has anticipated his result, which must be $11n_7$.)

Solution by A. B. Farnell, University of Colorado. Assuming a solution of the difference equation

$$m_{i+2} - m_{i+1} - m_i = 0$$

of the form $m_i=cq^i$, it is found that the general solution is given by

$$m_i = c_1p^i + c_2q^i,$$

where $p=(1+\sqrt{5})/2$ and $q=(1-\sqrt{5})/2$ are the roots of $x^2-x-1=0$. Hence, remembering that $q=-1/p$ and $p^2-p-1=0$, we have

$$\begin{aligned}\sum_{i=1}^{4k-2} n_i &= c_1p(p^{4k-2}-1)/(p-1) + c_2q(q^{4k-2}-1)/(q-1)\\ &= c_1p^{2k+1}(p^{2k-1}+q^{2k-1}) + c_2q^{2k+1}(q^{2k-1}+p^{2k-1})\\ &= (p^{2k-1}+q^{2k-1})(c_1p^{2k+1}+c_2q^{2k+1}) = (p^{2k-1}+q^{2k-1})n_{2k+1},\end{aligned}$$

which is the desired result with

$$r_k = p^{2k-1} + q^{2k-1} = 2^{1-2k}[(1+\sqrt{5})^{2k-1} + (1-\sqrt{5})^{2k-1}]$$
$$= 2^{2-2k}\sum_{s=0}^{k-1}\binom{2k-1}{2s}5^s.$$

Also solved by Joseph Bram, H. C. Davis, M. S. Klamkin, Roger Lessard, Ou Li and Hsien-yü-Hsü, J. F. Locke, Leo Moser, S. T. Parker, J. Rosenbaum, Guy Stevenson, F. Underwood, and the Proposer.

Editorial Note. Stevenson shows that $r_k = N_{2k-1}$, where N_j is the particular case of n_j resulting when $a=1$, $b=3$. Thus r_k satisfies the difference equation $r_{k+2} = 3r_{k+1} - r_k$ with $r_1 = 1$, $r_2 = 4$ (and $r_3 = 11$ as in the cited example.)

Rational Functions of $\cos^n\theta$ and $\sin^n\theta$

4273 [1947, 550]. *Proposed by I. S. Cohen, University of Pennsylvania*

Prove that for any positive odd integer n, $\cos\theta$ and $\sin\theta$ are rational functions of $\cos^n\theta$ and $\sin^n\theta$ with rational coefficients. Find explicit expressions in the case $n=3$.

Solution of the Proposer. Denote $\cos\theta$ and $\sin\theta$ by c and s respectively, and place $p=c^n$, $q=s^n$. We then have $c^2+s^2=1$ and must prove that c and s are contained in the field $F(p, q)$ of all rational functions of p and q with coefficients in the field F of rational numbers. In other words we must show that

$$F(c, s) = F(p, q). \tag{1}$$

We note first that $F(p) \subseteq F(c) \subseteq F(c, s)$. Now clearly

$$[F(c, s) : F(c)] = 2,$$

where the brackets denote, as usual, the relative field degree. Also we have $[F(c) : F(p)] = n$, so that $[F(c, s) : F(p)] = 2n$. On the other hand, $F(p) \subseteq F(p, q) \subseteq F(c, s)$, so that (1) is now equivalent to

$$[F(p, q) : F(p)] = 2n. \tag{2}$$

To prove this, let $f(x, y)$ be the irreducible polynomial over F such that $f(p, q) = 0$, and let C be the curve $f(x, y) = 0$. Thus C is the curve whose general point is (p, q)—that is, it has the parametrization $x=c^n$, $y=s^n$. It should be noted that the real part of this curve satisfies the equation $x^{2/n}+y^{2/n}=1$, and is thus a generalization of the familiar hypocycloid of four cusps.

We determine the degree of $f(x, y)$ by finding the number of intersections of C with the line $y=0$. There is no intersection at infinity, for p and q infinite implies the same for c or s, hence $p/q = (c/s)^n = \pm i$, so that C meets the line at infinity only in the circular points. The line $y=0$ thus meets C only in the points $(\pm 1, 0)$. To find the intersection multiplicity at $(1, 0)$, we seek to introduce s as local parameter. Thus

$$q = s^n, \qquad p = c^n = (1-s^2)^{n/2} = 1 - \tfrac{1}{2}ns^2 + \cdots.$$

Since n is odd it follows that s is, in fact, the local parameter. (If n were even, it would be s^2.) Thus q vanishes to the nth order, and the intersection multiplicity at $(1, 0)$ is n. Since it is the same at $(-1, 0)$, the total intersection multiplicity is $2n$, which is therefore the order of C. The degree of $f(x, y)$ is therefore $2n$, and since the point at infinity on the y-axis is not on C the degree of $f(x, y)$ in y is also $2n$. Thus q satisfies over $F(p)$ an irreducible equation of degree $2n$, and (2) follows.

Editorial Note. The Proposer employs results from the theory of algebraic curves to obtain an explicit form for $c \equiv g(p, q)/h(p, q)$, where g and h are polynomials. The following elementary procedure, suggested by C. R. Phelps, is quicker and produces the result in simpler form.

We have

$$s^2 = 1 - c^2, \quad p = c^3, \quad q = s^3,$$
$$q^2 = (1 - c^2)^3 = 1 - 3c^2 + 3pc - p^2,$$
$$cq^2 = c - 3p + 3pc^2 - p^2c.$$

Considering the last two equations as linear equations in c and c^2 and solving for c, we find

$$c = p(2 + p^2 + q^2)/(2p^2 - q^2 + 1).$$

s is found by interchange of p and q. The extension of the method for larger values of n is obvious.

Divergent Series

4278 [1948, 34]. *Proposed by Peter Ungar, Budapest, Hungary*

Construct two divergent series, $\sum a_k$ and $\sum b_k$ with $a_1 \geqq a_2 \geqq \cdots \geqq 0$, $b_1 \geqq b_2 \geqq \cdots \geqq 0$, but such that if $c_k = \min(a_k, b_k)$, $c_k > 0$, then $\sum c_k$ is convergent.

Solution by A. Novikoff, Stanford University. We consider any two series of positive terms, one convergent and one divergent, whose general terms, $C(n)$ and $D(n)$ respectively, tend monotonically to zero and such that $D(n) > C(n)$.

Since $\sum D(n)$ is divergent, corresponding to each integer n we can find another integer $\phi(n) > n$ defined by the relation

$$\sum_{j=n}^{\phi(n)} D(j) \geqq \epsilon,$$

where ϵ is any fixed positive number. We now define the sequence of integers $\{n_k\}$ recursively by the conditions $n_1 = 1$, $D(n_{k+1}) < C(\phi(n_k))$. It is clear that $n_k < \phi(n_k) < n_{k+1}$.

Now a solution to the problem is given by the sequences:

$$\{a_n\}; \quad C(n_1), C(n_1 + 1), \cdots, C(\phi(n_1)), D(n_2), D(n_2 + 1), \cdots, D(\phi(n_2)),$$
$$C(n_3), \cdots, C(\phi(n_3)), D(n_4), \cdots;$$

$$\{b_n\};\quad D(n_1), D(n_1+1), \cdots, D(\phi(n_1)), C(n_2), C(n_2+1), \cdots,$$
$$C(\phi(n_2)), D(n_3), \cdots, D(\phi(n_3)), C(n_4), \cdots.$$

It is clear that $\sum a_n$ and $\sum b_n$ both diverge, since each contains infinitely many stretches of terms adding up to ϵ or more, and that $\sum c_n$, where $c_n = \min(a_n, b_n)$ is convergent by comparison with $\sum C(n)$.

Also solved by Joshua Barlaz, M. T. Bird, Colin Blyth, M. K. Fort, Jr., Michael Golomb, William Gustin, H. J. Hamilton, Fritz Herzog, M. S. Klamkin, N. H. Kuiper, Norman Miller, S. T. Parker, George Piranian, W. Seidel, Albert Wilansky, F. E. Wood, and the Proposer.

Editorial Note. Novikoff extended his solution to show how to construct k divergent series of monotonically decreasing positive terms such that the minorant of all k series converges, but the minorant of every set of $k-1$ of the series diverges.

Many of the solvers showed that any convergent series of monotonically decreasing terms could be taken for $\sum c_n$ .The Proposer proved that, given divergent $\sum a_n$, the necessary and sufficient conditions that divergent $\sum b_n$ can be constructed such that $\sum c_n$ converges is that $\overline{\lim}\,(1/na_n) = \infty$.

A Problem of Pursuit

4280 [1948, 100]. *Proposed by Joseph Rosenbaum, Milford School, Milford, Connecticut.*

The points $A_1, A_2, \cdots, A_n$ are pursuing one another cyclically, the speed of A_i being proportional to the distance A_iA_{i+1}. If at time $t=0$ the points are the vertices of a given polygon, find the paths described by the points.

Solution by J. B. Reynolds, Salt Lake City, Utah. The motion must satisfy the set of vector equations

$$(1)\qquad k_i \frac{d\mathbf{r}_i}{dt} = \mathbf{r}_{i+1} - \mathbf{r}_i,\ i = 1, 2, \cdots, n-1$$
$$k_n \frac{d\mathbf{r}_n}{dt} = \mathbf{r}_1 - \mathbf{r}_n,$$

where $\mathbf{r}_i$ is the radius vector to the point A_i and k_i are constant.

The cartesian coördinates of A_i must then satisfy the identities

$$(2)\qquad Lx_i = x_i, \qquad Ly_i = y_i,$$

in which L is the operator

$$(k_{i-1}D + 1)(k_{i-2}D + 1) \cdots (k_1D + 1)(k_nD + 1) \cdots (k_iD + 1),$$

where $D \equiv d/dt$. The equations (2) have the solutions

$$x_i = \sum_{j=1}^{n} c_i^j e^{\lambda_j t}, \qquad y_i = \sum_{j=1}^{n} d_i^j e^{\lambda_j t},$$

the λ_j being (distinct) roots of

$$(k_n\lambda + 1)(k_{n-1}\lambda + 1) \cdots (k_1\lambda + 1) = 1. \tag{3}$$

To evaluate the constants c_i^j, note that the relation

$$\sum_{j=1}^{n} k_i c_i^j \lambda_j e^{\lambda_j t} = \sum_{j=1}^{n} c_{i+1}^j e^{\lambda_j t} - \sum_{j=1}^{n} c_i^j e^{\lambda_j t}$$

must hold for all values of t. This implies that

$$c_{i+1}^j = (k_i\lambda_j + 1)c_i^j. \tag{4}$$

Let the initial values of the x_i be $x_1^0, x_2^0, \cdots, x_n^0$; then the relations

$$\sum_{j=1}^{n} c_i^j = x_i^0 \tag{5}$$

must hold.

The equations (5) together with the recurrence relations (4) are sufficient to determine the constants c_i^j. Analogous relations hold for the d_i^j.

In the particular case $k_1 = k_2 = \cdots = k_n = k$, equation (3) becomes $(k\lambda+1)^n = 1$ with $\lambda_j = (\omega_j - 1)/k$ where the ω_j are the nth roots of unity. In this case (4) becomes

$$c_{i+1}^j = \omega_j c_i^j. \tag{6}$$

Nothing in the solution thus far requires that the A_i be coplanar. In the three dimensional case, the analysis for z_i proceeds exactly as for x_i, y_i. Note also from (1) that $\sum x_i$, $\sum y_i$ remain constant so that the centroid of the set A_i is fixed.

Consider further the special case where the initial polygon is regular. For convenience let the vertices of the polygon coincide with the termini of the vectors representing the nth roots of unity in the complex plane. Let

$$\omega_1 = e^{2\pi i/n}, \qquad \omega_{i+1} = (\omega_1)^{i+1}, \qquad \omega_n = \omega_0 = 1, \tag{7}$$

and note that

$$\sum_{j=1}^{n} b_i^j = \sum_{j=1}^{n} (c_i^j + id_i^j) = x_i^0 + iy_i^0 = \omega_{i-1}. \tag{8}$$

The relations given in (6), (7) and (8) determine $b_1^1 = 1$, $b_i^j = 0$ $(j > 1)$, so that the final solution is

$$x_1 + iy_1 = e^{(e^{2\pi i/n} - 1)\, t/k}.$$

In polar coördinates,

$$\rho_1 = e^{-(1-\cos 2\pi/n)\, t/k}, \qquad \theta_1 = \frac{t}{k} \sin \frac{2\pi}{n}.$$

To obtain the equations of motion of A_{i+1}, add $2\pi/n$ to θ_i.

Also solved by the Proposer.

Integers with Digits 0 and 1

4281 [1948, 100] *Proposed by M. S. Knebelman, Washington State College*

Given an integer n. Show that an integer can always be found which contains only the digits 0 and 1 (in the decimal scale) and which is divisible by n. Is there an algorithm for finding the smallest such number?

Solution by Leo Moser, University of Manitoba. Consider the residues (mod n) of the n integers $(10^i-1)/9$, $i=1, 2, \cdots, n$. Either there is one of them which is 0, or two are the same in which case the residue of the difference of the corresponding numbers is 0. In either case we have an integer composed of a block of 1's followed by a block of 0's divisible by n. The same argument, using $(B^i-1)/(B-1)$, leads to the same result in any system of numeration, with base B.

We have thus proved that the required integer has no more than n digits Hence if we test 1, 10, 11, 100, 101, $\cdots$, $(10^n-1)/9$ for divisibility by n we will certainly find the smallest number of the required type in not more than 2^n such operations. Clearly the work will be considerably cut down by first constructing a table of residues of 10^i (mod n).

Also solved by Colin Blyth, D. H. Browne, P. A. Clement, Roger Lessard, and F. L. Miksa.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

A Source Book in Greek Science. By M. R. Cohen and I. E. Drabkin. New York, McGraw-Hill Book Co., 1948. 21+579 pages. $9.00.

A great part of the attractiveness of ancient science arises from the important fact that the reader can check its statements for himself. If an amateur astronomer wishes to repeat modern observations on the length of the year, he must have access to expensive and elaborate equipment; but to determine for himself that, as Hipparchus says, the estimate of 365 days is probably wrong by about a quarter of a day, he need only follow the advice of many ancient

authors, from the poet Hesiod onward for a thousand years; throughout this period he will find a rich association of poetry and science, impossible in modern times, when science has gone so far beyond the thoughts and observations of daily life. The study of such classics will reward him, in the words of Albert Jay Nock, with a "feeling of immense longevity." The situation is the same in mathematics. The amateur who is interested in such natural questions as whether the number of primes is infinite, or what was the original form of the basic theorems of geometry, need only turn the pages of the book under review.

There are nine divisions in the book: Mathematics, 89 pages, Astronomy 54, Mathematical Geography 39, Physics 170, Chemistry and Chemical Technology 22, Geology and Meteorology 20, Biology 73, Medicine 63, and Physiological Psychology 27. The present review is confined to the first and third of these sections, with the parenthetical remark that the other seven appear to be on an equally high level and some of them contain considerable mathematics.

The first section begins with a chapter from Proclus, on "Divisions of Mathematics, Pure and Applied"; and in suitable places appear other passages from the same writer, a fifth century Neoplatonist who wrote a valuable commentary on Euclid. There follow eleven paragraphs on the theory of numbers, including the infinitude of primes and the Eudoxian theory of irrationality. These and other passages are either quoted from standard translations or are translated for the first time by Dr. Drabkin; in each case they are accompanied by brief but excellent introductions and notes. There are three passages on Algebra; the second of these is the famous Cattle-Problem of Archimedes, and the third consists of problems from Diophantus, both determinate and indeterminate, including the one which gave rise to Fermat's well-known marginal note.

The passages on Geometry are numerous and some of them are of the highest interest. They begin with a full quotation of the valuable section in Proclus on the history of geometry up to the time of Euclid; there is a discussion, from Pappus, of analysis and synthesis; such matters as the parallel postulate and the method of exhaustion are presented in quotations of generous length; the three geometrical problems, squaring of the circle, duplication of the cube, and trisection of an angle, are adequately represented in 17 pages; there is an illuminating extract from the "Method" of Archimedes for solving problems of the integral calculus; then come passages from Apollonius on conic sections and the charming mathematical excursus from Pappus on the sagacity of bees; the section on Mathematics closes with a brief treatment of Greek trigonometry and mensuration.

The section on Mathematical Geography begins with a general discussion from Aristotle on the shape and size of the earth, followed by an account of the simple measurements of Eratosthenes whereby he determined the polar circumference as being about 23,300 miles. The section closes with various accounts, both technical and popular, on the art of map-making, with excellent illustrations of maps of the ancient world according to Hecataeus (*ca.* 500 B.C.),

Eratosthenes (*ca.* 250 B.C.) and Ptolemy (*ca.* 150 A.D.); the methods given for representing the inhabited earth on a plane surface rank high in mathematical interest.

From this partial account it will be clear that the Source Book is heartily recommended to any reader who is interested, from whatever point of view, in the history of Greek mathematics. Its only fault, one may say, is that it is not more complete. Nothing is quoted from Apollonius except the prefaces to some of his books and Archimedes is represented in an almost equally meager way. The reader may well exclaim: "then how can you speak of a source book in Greek Mathematics?" But there are many extenuating circumstances. As we read in the preface, "the availability to English readers of the late Sir Thomas L. Heath's *History of Greek Mathematics* and of Ivor Thomas's translations . . . has prompted the editors to devote less space to this field in order to give additional space to fields in which the material is not so readily available." The difficulty is fundamental to all classical scholarship; if less material were available, a scholar could take account of all of it; and if more, then he would not be expected to. The present editors have dealt with the dilemma in a skilful way.

The format, printing and paper of the book are excellent; misprints are rare and unimportant, there being one such misprint, for example, on p. 562; the notes and introductory material will be a delight to the reader; and finally, the translations are on the same plane as the other features of the book.

With respect to these translations, the reviewer would like to make one query. In Dr. Drabkin's version of the passage from Pappus (p. 80), which deals with solids of revolution, reference is made only to the volume of such solids. But the Greek words seem to allow reference to both volume and surface; i.e. ὁ μὲν τῶν τελείων ἀμφοιστικῶν λόγος means "the ratio of the figures (whether volumes or surfaces) generated by complete revolution," and ἔκ τε τῶν ἀμφοισμάτων refers to the plane figures (whether areas or circumferences) which are revolved. In other words, is there not reference here to the two theorems now known as theorems of Pappus, and not to just one of them?

A minor point in translation is this: would it not be better to use *logistic* throughout, as in the note on p. 4, to translate λογιστική (the art of calculation), rather than *logistics*, as on p. 4 and frequently?

Taken together with a projected work on Medieval Science, the present volume forms a welcome addition to the other Source Books of the series, each of which is devoted to the progress of a single science from 1400 to 1900. Also, we are assured that the period since 1900 will not be neglected. In a special preface by the general editor of the series, we read: "a single volume, containing the most important contributions of the major sciences from 1900 to 1950, is planned for publication about 1960, and a similar volume each half century thereafter indefinitely." Such laudable far-sightedness is presumably the result of classical reading; let us hope that the present volume on Greek science will produce in all its readers a similar "feeling of longevity."

S. H. GOULD

A Collection of Papers in Memory of Sir William Rowan Hamilton. Scripta Mathematica Studies, No. 2. New York, Scripta Mathematica, 1945. 82 pages. $1.00.

The material in this little volume can be separated into three categories. First there are discussions of the life of Hamilton: *Sir William Rowan Hamilton* by David Eugene Smith, which was taken from Smith's *Portraits of Eminent Mathematicians, with Brief Biographical Sketches*; and *The Life and Early Work of Sir William Rowan Hamilton* by J. L. Synge.

A second group of essays is concerned with the scientific work of Hamilton: the paper already mentioned by Synge; *Algebra's Debt to Hamilton* by C. C. MacDuffee; *An Elementary Presentation of the Theory of Quaternions* by F. D. Murnaghan; *Hamilton's Work in Dynamics and Its Influence on Modern Thought* by H. Bateman; *Hamilton's Contribution to Mechanics* by Edwin B. Wilson; *The Constancy of the Velocity of Light* by Vladimir Karapetoff. Karapetoff's paper, the longest in the collection, contains no mention of Hamilton or his work. Wilson's paper is an excerpt from a letter to the editor of *Scripta Mathematica*, less than a page in length. Bateman's paper has 106 items in the bibliography, and each of these references is related to the essay.

The third category of material is a number of personal items: a portrait, two poems by Hamilton, facsimiles of two pages from his notebooks, and an announcement concerning an Irish postage stamp issued on the occasion of the centenary of Hamilton's discovery of quaternions.

Quaternions, of course, are prominent in many of the papers. MacDuffee treats them as quadruples of real numbers, as did Hamilton, whereas Murnaghan gives a representation of quaternions as special skew-symmetric matrices of order 4.

Several of the writers give their views on the relative merits of quaternions and Hamilton's contributions to applied mathematics. And Synge compares aspects to Hamilton's work with that of Lagrange and Jacobi. These discussions help make it a lively and entertaining, as well as informative, volume.

IVAN NIVEN

Elementary Statistical Analysis. By S. S. Wilks. Princeton University Press, 1948. 11+284 pages. $2.50.

This book is intended for a one semester elementary course in statistics. As the author states in the preface, he has been teaching this material as an introduction to statistics to students in all fields of application. In accordance with such a purpose only a very modest mathematical background is presupposed, essentially just the concepts of derivative and integral, such as a student might acquire in a one term calculus course.

It is clear that on this basis a complete treatment of much of the theory of statistical inference is not possible. The common way out of this difficulty is to make such a text a collection of methods, stated as a set of directions, together

with illustrations of their use. The present book—breaking away from this unfortunate tradition—has as its goal the understanding of statistics rather than manipulative proficiency. As the foundation for such understanding the elementary (discrete) theory of probability is developed. This includes in particular the more important theorems concerned with the notions of addition, multiplication and complementation of events, and of expectation and variance of a random variable. This material, together with a discussion of the binomial and Poisson distributions and with some extensions to the continuous case, forms the central part of the book, taking up about one half of the space.

Against this background it is possible to give a satisfactory account of the fundamentals of statistical inference. Confidence intervals and tests of significance are treated, and the notions are applied to the problems of inference on the basis of a large sample, concerning a single mean and the difference of two means. The case of small samples from normal distributions is also considered, and there is a brief chapter on testing for randomness. In this part certain results from probability theory have to be stated without proof. However, the resulting loss of understanding does not seem to be too serious since the content of the theorems can be explained satisfactorily even though the proofs cannot be given. The book also contains three chapters on descriptive statistics, and a chapter on correlation and regression for two variables together with a discussion of the method of least squares. Throughout the text, each chapter is followed by a set of exercises.

A few minor points may be raised:

(i) One wonders why in the theorem asserting linearity of the operation of expectation, independence of the variables is stressed. The theorem is true, useful, and essentially no harder to prove when the variables are dependent.

(ii) It seems a pity that the power of a test is never mentioned explicitly and that in connection with estimation no mention is made of the justification of the method of least squares from the point of view of unbiased estimation.

(iii) A section on the χ^2-distribution and some of its applications would fit well within the framework of the book, and would be of great value to most of its users.

By providing an elementary course that can be taught to students in many fields of application, and in which the objective is the teaching of concepts rather than rules, Professor Wilks has made an important contribution to elementary instruction in statistics. One may also hope that his book will encourage the centralization, within universities, of elementary statistical teaching.

E. L. LEHMANN

Les Equations Differentielles de la Technique. (Cours de Mathematiques Appliquees de L'Ecole Polytechnique de L'Universite de Lausanne). By Charles Blanc. Neuchatel, Editions du Griffin, 1947. 311 pages, unbound. Fr. s. 29.50.

This book is a text on the advanced calculus level for engineering students.

Ordinary and partial linear differential equations are studied with many examples from mechanics, electricity, and heat. Most of the equations considered have constant coefficients. The book is well written. The first part is on ordinary differential equations and includes a discussion of the steady state and transient solutions with electrical and mechanical examples and use of the Laplace transform. A chapter on boundary value problems, the Green's function and Fourier series with simple examples complete Part One.

Part Two takes up the d'Alembert equation, the heat equation and Poisson's equation where examples from electricity and mechanics are considered and the treatment includes the Laplace transform, propagation of waves, fundamental singular solutions, and Green's theorem and functions.

Part Three contains a chapter on the calculus of variations, one on elliptic integrals and one on Bessel functions.

NORMAN LEVINSON

NEW BOOKS RECEIVED

Analytic Geometry. By R. Robinson. New York, McGraw-Hill Book Co., 1949. 10+147 pages. $2.25.

Intermediate Algebra for Colleges. By P. R. Rider. New York, The Macmillan Co., 1949. 12+242 pages. $2.75.

Solid Geometry. By J. S. Frame. New York, McGraw-Hill Book Co., 1948. 10+339 pages. $3.50.

First Year College Mathematics with Applications. By P. H. Daus and W. M. Whyburn. New York, The Macmillan Co., 1949. 16+495 pages. $5.00.

Infinitesimalrechnung. Vol. I (3rd Edition), Vol. II (2nd Edition). By H. Behnke. Munster, Aschendorffsche, 1947 and 1948. 5+309 pages and 5+353 pages.

Introduction to Complex Variables and Applications. By R. V. Churchill. New York, McGraw-Hill Book Co., 1948. 6+216 pages. $3.50.

Problem Book in The Theory of Functions. Vol. I. By K. Knopp. Translated by L. Bers. New York, Dover Publications, 1949. 8+126 pages. $1.85.

Lezioni di Geometria Analitica e Projettiva. Second Edition. By G. Fano and A. Terracini. Torino, Paravia, 1948. 8+642 pages.

Niet-Euklidische Meetkunde. Second Edition. By J. C. H. Gerretsen. Gorinchem, Noorduijn's, 1949. 11+212 pages.

Numerical Calculus, By W. E. Milne. Princeton University Press, 1949. 10+393 pages. $3.75.

Practical Analysis. By F. A. Willers. Translated by R. T. Beyer. New York, Dover Publications, 1948. 10+422 pages. $6.00.

Fluid Dynamics. By V. L. Streeter. New York, McGraw-Hill Book Co., 1948. 11+263 pages. $5.00.

Computation Curves for Compressible Fluid Problems. By C. L. Dailey and F. C. Wood. New York, John Wiley and Sons. 1949. 10+33 pages, plus tables. $2.00.

Psychological Statistics. By Q. McNemar. New York, John Wiley and Sons, 1949. 8+364 pages. $4.50.

Archibald Henderson, The New Crichton. Edited by S. S. Hood. New York, Beechhurst. 1949. 18+252 pages. $5.00.

Integraltafel. Part I. By W. Grobner and N. Hofreiter. Vienna, Springer-Verlag, 1949. 8+166 pages. $5.40.

Theorie und Anwendung der Unendlichen Reihen. Fourth Edition. By K. Knopp. Berlin, Springer-Verlag, 1947. 12+583 pages.

Advances in Applied Mechanics. Vol. I. Edited by R. von Mises and T. von Karman. New York, Academic Press, Inc., 1948. 8+292 pages, $6.80.

Selected Techniques of Statistical Analysis. By The Statistical Research Group, Columbia University. New York, McGraw-Hill Book Co., 1947. 14+473 pages. $6.00.

An Essay Toward a Unified Theory of Special Functions. By C. Truesdell. Princeton University Press, 1948. 182 pages. $3.00.

Tables of Bessel Functions of the First Kind of Orders Forty through Fifty-one. (Annals of the Computation Laboratory of Harvard University, No. 9). Cambridge, The Harvard University Press, 1948. 14+620 pages. $10.00.

Principles of Mechanics. Second Edition. By J. L. Synge. New York, McGraw-Hill Book Co., 1949. 16+530 pages. $5.00.

Fundamental Theory. By A. S. Eddington. Cambridge, At the University Press, 1946. 8+292 pages. $6.00.

The Psychology of Invention in the Mathematical Field. Revised Edition. By J. Hadamard. Princeton University Press, 1949. 9+145 pages. $2.50.

Children Discover Arithmetic. By C. Stern. New York, Harper and Brothers, 1949. 24+295 pages. $4.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1947–48

Pi Mu Epsilon, Hunter College, New York

The theme of the Hunter College Chapter of *Pi Mu Epsilon* for the Fall term of 1947 was *Mathematical paradoxes*. Among the topics discussed were:

Elementary paradoxes in algebra and geometry

Plane analytic geometry in the complex field

Cardinal numbers

Transformation of the affine, euclidean, and rigid motion groups

Transfinite numbers and paradoxes of the infinite, by Sylvia Schlachter and Suzanne Levine.

The guest speaker for the initiation dinner was Prof. Saunders MacLane of the University of Chicago, who spoke on: *Mathematical centers here and abroad.*

The theme for the Spring term was *Continued fractions*. Material presented dealt with:

Convergents
Rational numbers
Solution of diophantine linear equations
Irrational numbers
Irrational quadratic algebraic numbers
Infinite series

The numerical solution of algebraic equations by continued fractions, by Anna Lemont.

The speaker for the Spring initiation was Prof. Samuel Eilenberg of Columbia University, whose talk entitled *Some aspects of topology* concluded an evening of relaxation and enjoyment.

Kappa Mu Epsilon, Alabama College

Papers presented during the year included:

Pythagoras and the Pythagoreans, by Lida True
Some topics from the theory of numbers, by Dr. Rosa Lea Jackson
Regular polygons and how they grow, by Margaret O'Gwynn
Waves and vibrations, by Frances Jones
Mathematics and the physical sciences, by Virginia Havens.

The first meeting of the year was a party to which all upper classmen majoring in Mathematics were invited. In the Spring, eight new members were initiated. A banquet followed the initiation ceremony.

Officers elected for the year 1948–49 are: President Frances Kelly; Vice-President, Margaret O'Gwynn; Secretary, Doris Williamson; Treasurer, Betty Lou Wilson.

Mathematics Society, Cooper Union

The Cooper Union *Mathematics Society* enjoyed the following program of student papers during 1947–48:

The calculus of finite differences and difference equations, by D. Jagerman
Convergence tests, by H. Hockstadt
Indeterminate equations and their applications, by M. Stern
Pythagorean primitives, by B. Lowe
Introduction to the calculus of variations, by M. Rosenfeld
Recurring series, by D. Jagerman
The Laplace transform, by R. Cox
Theory and application of the D operator, by H. Hochstadt
Congruences, by A. Kahn
Determinants and linear equations, by M. Stern

Matrix algebra, by D. Jagerman
Non-Euclidean geometry, by E. Wachspress.

The officers for 1948–49 are: President, D. Jagerman; Vice-President, H. Hochstadt; Secretary, M. Stern; Treasurer, M. Rosenfeld; Faculty Adviser, Prof. J. N. Eastham.

Kappa Mu Epsilon, The College of Wooster

The *Ohio Beta* Chapter of *Kappa Mu Epsilon*, inactive during the war, resumed activities at the close of the school year 1946–47. There are at present twenty-two undergraduate members. The following talks by members of the faculty and undergraduate body were presented during 1946–48:

Topology, by Prof. M. P. Fobes
Einstein's theory of relativity, by Prof. R. J. Stephenson
Some interesting curves, by the Chapter Initiates
Foundations of mathematics, by Dorothy Renzema
Non-Euclidean geometry, by Duncan McCune
Mathematical fun, by the Chapter Initiates.

The Wilson Mathematical Prize for the year was won by Margaret Hagen.

The officers for 1948–49 are: President, Robert Nethercut; Vice-President, Robert Shaffer; Treasurer, John Richardson; Secretary, Margaret Herr; Faculty Sponsor, Prof. M. P. Fobes.

Pi Mu Epsilon, Kansas State College

Talks presented to the *Kansas Beta* Chapter of *Pi Mu Epsilon* included:
Nomographs, by Prof. C. E. Pearce
Approximation of functions by integral means, by Dr. P. M. Young
Summability of series and integrals, by Dr. S. T. Parker.

Newly elected officers are: Director, Dr. P. M. Young; Vice-Director, Robert Cell; Secretary, Virginia Chatelain; Treasurer, Jack Northam.

The Zeno Club, Alfred University

The *Zeno Club* of Alfred University held semi-monthly meetings, with a program and discussion, followed by a social period with refreshments. Papers presented by the faculty and others were:

Nomograms, by Dr. C. E. Rhodes
Boolean algebra, by Prof. W. V. Nevins
Probabilities in genetics, by Prof. John Freund
Transfinite cardinal numbers, by Ralph Jordan
Number system to the base 2, by Leslie Shershoff
Euclidean algorithm, by Joan Berkman
Hyperbolic functions, by Stanley Graf
Topology, by Lewis Butler
Segment functions, by Prof. John Freund
Maxima and minima, by Irwin Miller
Cryptography, by Ralph Beals
Divisibility of numbers, by Dean A. E. Whitford.

The club's major projects of the year included the presentation of several books to the mathematics section of the Alfred University Library, and the sponsoring of a campus-wide competitive examination in calculus. The prizes given for this were: ten dollars, a season movie ticket, and a calculus book.

Officers for the year 1947–48 were: President, Ralph Jordan; Vice-President, Joan Berkman; Secretary-Treasurer, Mary Elizabeth Van Norman.

Pi Mu Epsilon, Carnegie Institute of Technology

On November 20, 1947, the Carnegie Institute of Technology *Mathematics Club*, which was founded by Prof. J. B. Rosenbach in 1938, was installed as the *Pennsylvania Epsilon* Chapter of *Pi Mu Epsilon*. The installation officers were Prof. Tomlinson Fort, Director-General and Prof. J. S. Gold, Secretary-Treasurer General. Prof. J. J. Stoker of New York University, an alumnus and former faculty member of Carnegie Institute of Technology, gave the installation address on the subject *Water waves on sloping beaches.*

A group of twenty members of the chapter visited the Westinghouse Electric Corporation in East Pittsburgh in order to witness a demonstration of the Westinghouse electric analog computer.

Prof. J. L. Synge was the guest of honor and principal speaker at the annual initiation and banquet. His paper, which will appear in the Carnegie Technical, was on the topic *Mathematics and Science.*

Four additional meetings were held during the year at which the following papers were presented:

Finite differences and statistics, by Mr. H. J. Weiss

The four-color problem, by Mr. C. G. Maple

Horn angles, Mr. G. H. F. Gardner

The Westinghouse analog computer, by Mr. D. L. Whitehead, of the Westinghouse Electric Corporation.

The officers for the year 1947–48 were: Director, D. L. Wallace; Vice-Director, D. R. Harris; Secretary, R. T. Siegel; Treasurer, E. P. King; Faculty Adviser, Prof. J. B. Rosenbach.

Mathematics Club, Lander College

The *Mathematics Club* of Lander College holds six regular open meetings during the school year. A social hour at which refreshments are served follows each program.

The 1947–48 programs, given by students and faculty, consisted of games, mathematical contests and puzzles, and papers on:

The life and mathematical achievements of Rene Descartes

A review of Eddington's "Space, Time, and Gravitation"

History of the development of the atomic theory.

The 1948–49 officers are: President, Odel Black; Vice-President, Emmilene Talbert; Secretary-Treasurer, Frankie Sue Dickerson; Sponsor, Prof. Mary Pettus.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

CANADIAN MATHEMATICAL CONGRESS

The second seminar of the Canadian Mathematical Congress will be held on August 16–September 10, 1949 at the University of British Columbia, Vancouver. The seminar will resemble in structure the 1947 seminar of Toronto, its arranged program including: (a) Courses of eight lectures each on research level; (b) Seminars (discussion groups); (c) Instructional courses of fifteen or more lectures at first year graduate level. The topics are Applied Mathematics and Related Fields of Pure Mathematics.

Courses (a) will be as follows: P. A. M. Dirac, University of Cambridge, England, dynamical theory of fields, classical and quantum; Antoni Zygmund, University of Chicago, topics connected with Fourier series; Laurent Schwartz, University of Nancy, theory of distributions and applications to ordinary and partial differential equations; and one other course by a distinguished applied mathematician.

Under (b) there will be two seminars, one in applied and one in pure mathematics under the direction respectively of Leopold Infeld, University of Toronto, and Gabor Szegö, Stanford University.

Instructional courses (c) will be as follows: M. S. Knebelman, Washington State College, topics from differential geometry; W. H. McEwen, University of Manitoba, the Sturm-Liouville problem; A. F. C. Stevenson, University of Toronto, electromagnetic theory; G. M. Volkoff, University of British Columbia, introduction to quantum mechanics.

The Second Canadian Mathematical Congress will be held on September 5–10, 1949 at the University of British Columbia, Vancouver. Its program is expected to include general lectures by Professors Dirac, Schwartz, Szegö, Zygmund. There will be a short lecture series by R. L. Jeffery, Queen's University, Kingston, pertaining to recent researches in the theory of integration of real functions, and by Borge Jessen, University of Copenhagen, on mean motion and almost periodic functions; A. W. Conway, Institute of Advanced Studies, Dublin, will give a short series of technical lectures together with a special lecture on W. R. Hamilton. There will be shorter research papers and discussions on research and graduate work as well as secondary school mathematics.

Various receptions and excursions are being arranged in connection with both seminar and congress. Registration fees for seminar and congress are $10 and $2, respectively. Accommodations are available at the University of British Columbia at reasonable rates.

Members of the Mathematical Association of America are cordially invited. For further information write to: Canadian Mathematical Congress, Engineer-

ing Building, McGill University, Montreal, Canada. For accommodations apply to: Professor R. D. James, Department of Mathematics, University of British Columbia, Vancouver, Canada.

INSTITUTE FOR MATHEMATICS TEACHERS

The ninth annual session of the Institute for Mathematics Teachers will be held at Duke University, August 8–19, 1949. The general theme of the Institute is "Mathematics at Work" (Arithmetic through Calculus). The program includes twenty-two lecturers from industry, business, science, and teachers of recognition. In addition to the lectures there will be nine study groups under the leadership of teachers with outstanding ability. The work of the Institute centers around the Mathematics Laboratory being established at Duke University.

The registration fee is $10.00. A complete program may be secured by writing to the Director, Professor W. W. Rankin, Mathematics Department, Duke University, Durham, North Carolina.

SYMPOSIA ON NUMERICAL ANALYSIS

The National Bureau is planning two symposia on the effective utilization of automatic digital computing machinery to be held in June, 1949 at the Bureau's Institute for Numerical Analysis in Los Angeles, California. Symposium I will pertain to construction and applications of conformal maps. Tentative dates for this symposium are June 24–25. Symposium II will cover probability methods in numerical analysis. This symposium is being arranged jointly by the Institute for Numerical Analysis and the Rand Corporation, with the assistance of the Atomic Energy Commission. It is tentatively planned for June 27–29.

Those who would be interested in attending either of these symposia may obtain further information from Dr. J. H. Curtiss, Institute for Numerical Analysis, Los Angeles 24, California.

SALE OF BOOKS

A complete set of *Rendiconti del Circolo matematico di Palermo* has been offered for sale. There are sixty-two bound volumes and some supplements. Anyone interested should negotiate directly with Signora Anselmo Lidia, Via Alessio Narbone, 56, Palermo, Italy.

SUMMER COURSES

The following institutions announce advanced courses in mathematics for the summer of 1949:

Syracuse University. June 5 to August 12: mathematics of statistics; introduction to higher algebra; college plane geometry; introduction to modern mathematics; history of mathematics; methods and materials in mathematics education aids. July 25 to September 2: advanced calculus; mathematics of statistics; introduction to higher algebra II; fundamentals of analysis; higher

mathematics for engineers and scientists II; vector analysis; functions of a real variable; differential geometry.

University of Pittsburgh. June 13 to September 2. The following nine full year courses will be presented in the twelve weeks' session, the work of a first semester from June 13 to July 22 and of a second semester from July 25 to September 2; Professor Blumberg, advanced calculus; Professor Bompiani, geometry of partial differential equations; Professor Booth, materials of mathematical physics (first six weeks) and mathematical and physical bases of x-ray crystal structure determination (second six weeks); Professor Culver, differential equations, modern algebraic theories; Mr. Gettig, functions of a complex variable; Professor Laush, functions of a real variable; Professor Michalik, mathematical theory of statistics, mathematical theory of probability. In addition the following briefer courses will be given in a six weeks' session from June 27 to August 5: Professor Foraker, algebra for teachers, geometry for teachers; Professor Knipp, solid analytic geometry, theory of equations; Professor Taylor, recreational mathematics for teachers.

PERSONAL ITEMS

Professor J. R. Musselman, Western Reserve University, was appointed a representative of the Association at the meeting of the Board of Foreign Scholarships which was held at Cleveland on March 30, 1949.

Professor J. H. Taylor, George Washington University, was appointed a delegate of the Association at the inauguration of the Very Reverend Hunter Guthrie as President of Georgetown University on May 1, 1949.

Dr. Harish-Chandra, Institute for Advanced Study, and Dr. J. A. Jenkins of Harvard University have been awarded two of the 1949–50 Frank B. Jewett Fellowships by the American Telephone and Telegraph Company.

Massachusetts Institute of Technology announces the promotions of Associate Professor Norman Levinson and Eric Reissner to professorships and the appointment of G. W. Whitehead to an assistant professorship.

University of Washington announces that Professor B. L. Van der Waerden of the University of Amsterdam will be Visiting Professor for the Summer Quarter, 1949. He will give a course in algebraic geometry and one in operational methods in mathematical physics.

Professor O. W. Albert, chairman of the Department of Mathematics of the University of Redlands, has been granted a half year's leave of absence and is a guest of the University of North Carolina where he is engaged in study at the Institute of Statistics.

Mr. T. H. Bedwell of the University of South Dakota has received an appointment as Assistant Professor of Physics at Florida State University.

Mr. S. K. Bright of Vanderbilt University has been appointed to a professorship at Austin Peay State College.

Professor H. L. Dorwart, Washington and Jefferson College, has been appointed Seabury Professor of Mathematics and Natural Philosophy at Trinity

College, Hartford, Connecticut. He will succeed Professor H. M. Dadourian who will retire on June 30, 1949 after thirty years of service.

Dr. S. W. Hahn of the University of Michigan has been appointed to an assistant professorship at Wittenberg College, effective September 1, 1949.

Assistant Professor J. G. Herriot of Stanford University has been promoted to an associate professorship.

Mr. E. W. Marchand, University of Rochester, has accepted a position as physicist with the Eastman Kodak Company of Rochester.

Professor L. F. Ollmann, chairman of the Department of Mathematics of Hofstra College, has been appointed to direct Summer Session I, 1949.

Mr. W. B. Stovall, Jr. of the University of Florida has accepted a position as statistician with the Bureau of Vital Statistics, State Board of Health, Florida.

Associate Professor F. J. Taylor of the College of St. Thomas has been promoted to a professorship.

Mr. J. S. Thompson, vice chairman of the Board of McGraw-Hill Book Company, retired on April 1, 1949.

Miss Lona L. Turner has been appointed to an instructorship at the Chicago Undergraduate Division of the University of Illinois.

Mr. W. H. Coulter of Decatur, Illinois, died on February 16, 1949.

Assistant Professor J. H. Cross of Texas Technological College died on December 8, 1948.

President-emeritus R. L. Durham of Southern Seminary and Junior College died January 1, 1949.

Dr. T. M. Focke, dean emeritus of Case Institute of Technology, died on March 2, 1949 at the age of seventy-eight years.

Mr. G. H. Selleck, retired head of the Department of Mathematics, Phillips Exeter Academy, died on March 26, 1949.

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Official Reports and Communications

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CALENDAR OF FUTURE MEETINGS

Joint Meeting with American Society for Engineering Education, Troy, New York, June 20–21, 1949.

Thirty-first Summer Meeting, Boulder, Colorado, August 29–30, 1949.

Thirty-third Annual Meeting, New York City, December 30, 1949.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

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LOUISIANA-MISSISSIPPI, Centenary College, Shreveport, Louisiana, Spring, 1950

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MINNESOTA, University of North Dakota, Grand Forks, October 15, 1949

MISSOURI, Spring 1950

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NORTHERN CALIFORNIA, Berkeley, January 28, 1950

OHIO, April, 1950

OKLAHOMA, Oklahoma City, October 14, 1949

PACIFIC NORTHWEST, University of Washington, Spring 1950

PHILADELPHIA, Haverford College, November 26, 1949

ROCKY MOUNTAIN

SOUTHEASTERN, University of Florida, Gainesville, March, 1950

SOUTHERN CALIFORNIA, Immaculate Heart College, Hollywood, March 11, 1950

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